

Integrable lattices and their sub-lattices: from the discrete Moutard (discrete Cauchy-Riemann) 4-point equation to the self-adjoint 5-point scheme.

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Abstract

We introduce the sub-lattice approach, a procedure to generate, from a given integrable lattice, a sub-lattice which inherits its integrability features. We consider, as illustrative example of this approach, the discrete Moutard 4-point equation and its sub-lattice, the self-adjoint 5-point scheme on the star of the square lattice, which are relevant in the theory of the integrable Discrete Geometries and in the theory of discrete holomorphic and harmonic functions (in this last context, the discrete Moutard equation is called discrete Cauchy-Riemann equation). We use the sub-lattice point of view to derive, from the Darboux transformations and superposition formulas of the discrete Moutard equation, the Darboux transformations and superposition formulas of the self-adjoint 5-point scheme. We also construct, from algebro-geometric solutions of the discrete Moutard equation, algebro-geometric solutions of the self-adjoint 5-point scheme. We finally use these solutions to construct explicit examples of discrete holomorphic and harmonic functions, as well as examples of quadrilateral surfaces in \mathbb{R}^3 .

1 Introduction

One of the most important methods to generate integrable equations is to start with a fairly general integrable system and apply to it systematically symmetry reductions. This approach was, for instance, applied to the multicomponent Kadomtsev–Petviashvili hierarchy [15] and to the self-dual Yang-Mills system [33]. On the level of discrete equations, it was systematically used, for instance, to generate various integrable reductions of the multidimensional quadrilateral lattice equations (see, for example, [8]).

In this paper we propose a different, but equally relevant, procedure to generate integrable lattices. This procedure consists in constructing, from a given integrable lattice, a sub-lattice which inherits the integrability features of the original lattice. We consider, as

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illustrative example of this approach, the 4-point difference equation

$$\phi_{m+1,n+1} - \phi_{m,n} = g_{m,n}(\phi_{m+1,n} - \phi_{m,n+1}), \quad (1)$$

where $g_{m,n}$ and $\phi_{m,n}$ are functions: $\mathbb{Z}^2 \rightarrow \mathbb{C}$, and its sub-lattice, the self-adjoint 5-point scheme:

$$a_{\mu,\nu}\Psi_{\mu+1,\nu} + a_{\mu-1,\nu}\Psi_{\mu-1,\nu} + b_{\mu,\nu}\Psi_{\mu,\nu+1} + b_{\mu,\nu-1}\Psi_{\mu,\nu-1} = c_{\mu,\nu}\Psi_{\mu,\nu} \quad (2)$$

on the star of the square lattice, where $a_{\mu,\nu}$, $b_{\mu,\nu}$, $c_{\mu,\nu}$ and $\Psi_{\mu,\nu}$ are functions: $\mathbb{Z}^2 \rightarrow \mathbb{C}$, and where the standard notation $f_{m,n} = f(m,n)$ is often used throughout the paper.

We exploit the above fact deriving several properties of the 5-point system (2) from the corresponding (and simpler) properties of the 4-point system (1).

As it was shown in [27] (see also [7]), equation (1) is an integrable discretization (i.e., a discretization possessing Darboux transformations (DTs)) of the Moutard equation

$$\Phi_{,uv} = F\Phi \quad (3)$$

(the symbol $f_{,u}$ denotes partial differentiation: $f_{,u} = \partial f / \partial u$). As it was shown in [24], equation (2) is an integrable discretization of the elliptic (if $AB > 1$) equation

$$(A\Phi_{,u})_{,u} + (B\Phi_{,v})_{,v} = F\Phi. \quad (4)$$

The paper is organized as follows. In section 2 we establish the connection between the above difference equations (1) and (2). In sections 3 and 4 we use the above connection to derive the Darboux transformation (DT) and its superposition formula of the 5-point scheme (2) (in all its most distinguished gauge equivalent forms) from the the Darboux transformation and its superposition formula of the 4-point scheme (1). In section 5 we first construct algebro-geometric solutions of the 4-point scheme (1) and then the corresponding solutions of the 5-point scheme (2). Due to the interesting applications of equations (1) and (2) (discussed in the second part of this introduction), these explicit solutions provide examples of discrete holomorphic and harmonic functions and, at the same time, they generate quadrilateral surfaces in \mathbb{R}^3 .

Let us devote the remaining part of the introduction to the presentation of the interesting applications of equations (1) and (2) in differential and discrete geometry, and in the discrete holomorphic function theory. Equation (1) has a very long history. Bianchi constructed the superposition formula for the Moutard equation (3) in the following form [2]

$$\Phi^{(12)} - \Phi = \frac{\theta^1 \theta^2}{\sigma} (\Phi^{(1)} - \Phi^{(2)}), \quad (5)$$

where Φ , $\Phi^{(1)}$, $\Phi^{(2)}$ and $\Phi^{(12)}$ are four different general solutions of four different Moutard equations (3), while θ^1 , θ^2 are particular solutions of the Moutard equation satisfied by Φ , and σ is a potential connected to θ^1 , θ^2 via the conditions

$$\sigma_{,u} = \theta^1 \theta^2_{,u} - \theta^2 \theta^1_{,u}, \quad (6)$$

$$\sigma_{,v} = \theta^2 \theta^1_{,v} - \theta^1 \theta^2_{,v}. \quad (7)$$

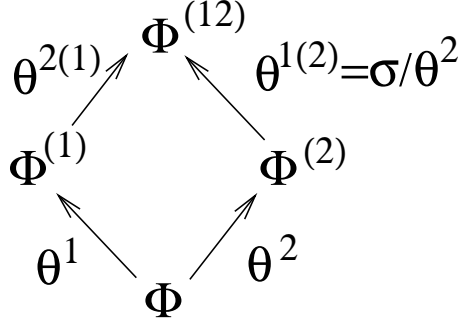


Figure 1: The Bianchi permutability diagram for the Moutard equation.

Using the key observation contained in [19], such superposition principle was interpreted in [27] as an integrable discretization (1) of the Moutard equation:

$$\Phi \rightarrow \phi_{m,n}, \quad \Phi^{(1)} \rightarrow \phi_{m+1,n}, \quad \Phi^{(2)} \rightarrow \phi_{m,n+1}, \quad \Phi^{(12)} \rightarrow \phi_{m+1,n+1}.$$

In addition, DTs and their superposition principle for the discrete Moutard equation (1) were also constructed in [27].

Solutions of the (real) Moutard equation (3) allow one to construct, via the Lelievre formulas [2], asymptotic nets on hyperbolic surfaces from their normal vector. Analogously, solutions of the (real) discrete Moutard equation (1) characterize normal vectors of asymptotic lattices [31, 3], which can be constructed via the discrete analogue of the Lelievre formulas [5].

Also for the elliptic version of equation (3), the well known Schrödinger equation

$$\Psi_{,xx} + \Psi_{,yy} = F\Psi \quad (8)$$

(which is equation (4) with $A = B = 1$), there exists a geometric interpretation within the theory of isothermally conjugate nets and the corresponding analogue of the Lelievre formulas [2]. The 5-point scheme (2), introduced in [24] as an integrable discretization of equation (4), describes the normal vector of a quadrilateral lattice in \mathbb{R}^3 , and this embedding is obtained via a suitable generalization of the Lelievre formulas [25].

It is remarkable that equations (1) and (2), which are basic equations in the recently developed theory of the Integrable Discrete Geometries, are also the basic equations of an integrable discretization of the theory of holomorphic and harmonic functions. Indeed the equations

$$\psi_{m+1,n+1} - \psi_{m,n} = i(\psi_{m+1,n} - \psi_{m,n+1}), \quad (9)$$

$$\psi_{\mu+1,\nu} + \psi_{\mu-1,\nu} + \psi_{\mu,\nu+1} + \psi_{\mu,\nu-1} = 4\psi_{\mu,\nu}, \quad (10)$$

undressed versions of equations (1) and (2), were introduced as basic objects of the discrete holomorphic and harmonic function theory in [13, 9]; they correspond to a natural discretization of, respectively, the $\bar{\partial}$ and Laplace operators on the square lattice, and the connection

between these two schemes was an important ingredient of the theory. In [10], this theory was generalized to rhombic lattices; on the level of the 4-point scheme, it corresponds to the discrete Cauchy-Riemann equation (1) with a nontrivial pure imaginary potential $g_{m,n}$. In [21] this theory was extended to discrete Riemann surfaces and, in particular, the 4-point scheme (1) and the affine form

$$a_{\mu,\nu}(\psi_{\mu+1,\nu} - \psi_{\mu,\nu}) + a_{\mu-1,\nu}(\psi_{\mu-1,\nu} - \psi_{\mu,\nu}) + b_{\mu,\nu}(\psi_{\mu,\nu+1} - \psi_{\mu,\nu}) + b_{\mu,\nu-1}(\psi_{\mu,\nu-1} - \psi_{\mu,\nu}) = 0 \quad (11)$$

of the self-adjoint 5-point scheme (2), which corresponds to a discretization of the harmonic equation, were studied on the Riemann surface. A theory of Dirac operators and the Green's function for planar graphs with rhombic faces were constructed in [16]. The connection between the above discrete complex function theory and the discrete complex function theory [32] based on the integrable cross-ratio equation [26] was established in [4], where a multidimensional generalization of the discrete complex function theory was also considered.

We remark that there exists another natural integrable discretization of elliptic operators on the plane, based on the self-adjoint scheme on the star of a regular triangular lattice [29], resulting in a 7-point scheme. One of the important features of this approach is that this operator is factorized in terms of 3-point operators living on the same lattice. A discrete complex function theory based on these operators, including the discrete analogue of the Cauchy kernel, has been recently developed in [11].

2 The 5-point scheme as a sub-lattice of the 4-point scheme

Consider a lattice, i.e. a map $\mathbf{x} : \mathcal{D} \rightarrow V$ from a grid \mathcal{D} to a linear space V satisfying a certain equation $E[\mathbf{x}] = 0$ (the lattice equation). Consider a subgrid $\mathcal{D}' \subset \mathcal{D}$; if one can construct, from the original lattice equation $E[\mathbf{x}] = 0$, a new equation $E'[\mathbf{x}'] = 0$, where \mathbf{x}' is the restriction of \mathbf{x} to the sub-grid \mathcal{D}' : $\mathbf{x}' = \mathbf{x}|_{\mathcal{D}'} : \mathcal{D}' \rightarrow V$, then \mathbf{x}' is a sub-lattice of \mathbf{x} and $E'[\mathbf{x}'] = 0$ is the associated sub-lattice equation.

Suppose that the original lattice (equation) be **integrable**; i.e., suppose that one can associate with it linear transformations enabling one to construct solutions from solutions (Darboux-type transformations), whose superposition exhibits permutability properties (the Bianchi permutability diagram). Since the infinite class of solutions generated in this way, once restricted, are also solutions of the sub-lattice, this sub-lattice (equation) will also be integrable. We remark that the dimensional reductions are simple examples of such a construction.

2.1 From the lattice to the sub-lattice

In this section we start with the general 4-point scheme

$$\alpha_{m,n}\varphi_{m+1,n+1} + \beta_{m,n}\varphi_{m+1,n} + \gamma_{m,n}\varphi_{m,n+1} + \delta_{m,n}\varphi_{m,n} = 0, \quad (12)$$

where α, β, γ and δ are \mathbb{C} -valued functions on $\mathcal{D} = \mathbb{Z}^2$, and we explore the possibility to construct, from it, a sub-lattice defined on the subset $\tilde{\mathcal{D}} = \mathbb{Z}_e^2$ (the even grid) of \mathbb{Z}^2 consisting of points (m, n) such that $m + n$ is even.

Proposition 1. *The general 4-point scheme (12) reduces to a 5-point scheme on the even grid if and only if the coefficients α, β, γ and δ satisfy the constraint*

$$\beta_{m,n}\alpha_{m-1,n}\delta_{m,n-1}\gamma_{m-1,n-1} = \gamma_{m,n}\delta_{m-1,n}\alpha_{m,n-1}\beta_{m-1,n-1}. \quad (13)$$

Proof. Consider the 4-point scheme (12) in the elementary square

$$Q_{m,n} = \{(m,n), (m+1,n), (m,n+1), (m+1,n+1)\}$$

and in its neighbouring squares $Q_{m-1,n-1}$, $Q_{m-1,n}$ and $Q_{m,n-1}$, and move to the LHS the terms in which φ is evaluated on the odd grid and to the RHS the terms in which φ is evaluated on the even grid:

$$\begin{aligned} \beta_{m,n}\varphi_{m+1,n} + \gamma_{m,n}\varphi_{m,n+1} &= -\alpha_{m,n}\varphi_{m+1,n+1} - \delta_{m,n}\varphi_{m,n}, \\ \alpha_{m-1,n}\varphi_{m,n+1} + \delta_{m-1,n}\varphi_{m-1,n} &= -\beta_{m-1,n}\varphi_{m,n} - \gamma_{m-1,n}\varphi_{m-1,n+1}, \\ \alpha_{m,n-1}\varphi_{m+1,n} + \delta_{m,n-1}\varphi_{m,n-1} &= -\gamma_{m,n-1}\varphi_{m,n} - \beta_{m,n-1}\varphi_{m+1,n-1}, \\ \gamma_{m-1,n-1}\varphi_{m-1,n} + \beta_{m-1,n-1}\varphi_{m,n-1} &= -\alpha_{m-1,n-1}\varphi_{m,n} - \delta_{m-1,n-1}\varphi_{m-1,n-1}. \end{aligned} \quad (14)$$

In order to construct a lattice equation not involving φ on the odd grid, it is sufficient to impose the zero determinant condition for the system (14), which coincides with the constraint (13). \square

Remark. The row vector solution \mathbf{v} of the homogeneous version of the system (14) reads:

$$\begin{aligned} \mathbf{v}_{m,n} = & (\gamma_{m-1,n-1}\alpha_{m-1,n}\alpha_{m,n-1}, -\gamma_{m,n}\gamma_{m-1,n-1}\alpha_{m,n-1}, \\ & -\alpha_{m-1,n}\beta_{m,n}\gamma_{m-1,n-1}, \alpha_{m,n-1}\delta_{m-1,n}\gamma_{m,n}) \end{aligned} \quad (15)$$

and the wanted sub-lattice equation is the solvability condition $(\mathbf{v}_{m,n}, \mathbf{F}_{m,n}) = 0$ of the Kronecker - Capelli theorem, where

$$\mathbf{F}_{m,n} = -(\alpha_{m,n}\varphi_{m+1,n+1} + \delta_{m,n}\varphi_{m,n}, \gamma_{m-1,n}\varphi_{m-1,n+1} + \beta_{m-1,n}\varphi_{m,n}, \beta_{m,n-1}\varphi_{m+1,n-1} + \gamma_{m,n-1}\varphi_{m,n}, \delta_{m-1,n-1}\varphi_{m-1,n-1} + \alpha_{m-1,n-1}\varphi_{m,n})^T \quad (16)$$

is the column vector representing the inhomogeneous term of the algebraic system. This solvability condition reads:

$$\begin{aligned} & \gamma_{m-1,n-1}\alpha_{m-1,n}\alpha_{m,n-1}(\alpha_{m,n}\varphi_{m+1,n+1} + \delta_{m,n}\varphi_{m,n}) - \\ & \gamma_{m,n}\gamma_{m-1,n-1}\alpha_{m,n-1}(\gamma_{m-1,n}\varphi_{m-1,n+1} + \beta_{m-1,n}\varphi_{m,n}) - \\ & \alpha_{m-1,n}\beta_{m,n}\gamma_{m-1,n-1}(\beta_{m,n-1}\varphi_{m+1,n-1} + \gamma_{m,n-1}\varphi_{m,n}) + \\ & \alpha_{m,n-1}\delta_{m-1,n}\gamma_{m,n}(\delta_{m-1,n-1}\varphi_{m-1,n-1} + \alpha_{m-1,n-1}\varphi_{m,n}) = 0. \end{aligned} \quad (17)$$

Proposition 2. *The four point scheme (12) with non-vanishing coefficients $\alpha, \beta, \gamma, \delta$ satisfying constraint (13) is gauge equivalent to the discrete Moutard (discrete Cauchy-Riemann) four point scheme (1) whose corresponding five point scheme is the affine self-adjoint five point scheme (11).*

Proof. Let the potential $\rho : \mathbb{Z}^2 \rightarrow \mathbb{C}$ be a solution of the system

$$\alpha_{m,n}\rho_{m+1,n+1} = -\delta_{m,n}\rho_{m,n}, \quad \beta_{m,n}\rho_{m+1,n} = -\gamma_{m,n}\rho_{m,n+1}, \quad (18)$$

whose compatibility condition is the constraint (13). Then the function $\psi : \mathbb{Z}^2 \rightarrow \mathbb{C}$

$$\psi_{m,n} = \varphi_{m,n} / \rho_{m,n}, \quad (19)$$

satisfies equation

$$\psi_{m+1,n+1} - \psi_{m,n} = i f_{m,n} (\psi_{m+1,n} - \psi_{m,n+1}), \quad (20)$$

with the potential

$$g_{m,n} = i f_{m,n} = \frac{\beta_{m,n} \rho_{m+1,n}}{\delta_{m,n} \rho_{m,n}} = -\frac{\gamma_{m,n} \rho_{m,n+1}}{\delta_{m,n} \rho_{m,n}}, \quad (21)$$

and equation (17) reduces to the 5-point scheme

$$\begin{aligned} a_{m,n}(\psi_{m+1,n-1} - \psi_{m,n}) + a_{m-1,n+1}(\psi_{m-1,n+1} - \psi_{m,n}) + \\ b_{m,n}(\psi_{m+1,n+1} - \psi_{m,n}) + b_{m-1,n-1}(\psi_{m-1,n-1} - \psi_{m,n}) = 0, \end{aligned} \quad (22)$$

with the functions $a_{m,n}$ and $b_{m,n}$ defined as follows

$$a_{m,n} = f_{m,n-1}, \quad b_{m,n} = \frac{1}{f_{m,n}}. \quad (23)$$

The connection between (20) and (22) we have established here can be easily verified directly considering the 4-point scheme (20) for the vector $\psi \in V$ in the elementary square $Q_{m,n}$ and in its neighbouring squares $Q_{m-1,n-1}$, $Q_{m-1,n}$, and $Q_{m,n-1}$:

$$\begin{aligned} \frac{1}{f_{m,n}}(\psi_{m+1,n+1} - \psi_{m,n}) &= i(\psi_{m+1,n} - \psi_{m,n+1}), \\ \frac{1}{f_{m-1,n-1}}(\psi_{m-1,n-1} - \psi_{m,n}) &= i(\psi_{m-1,n} - \psi_{m,n-1}), \\ f_{m,n-1}(\psi_{m+1,n-1} - \psi_{m,n}) &= -i(\psi_{m+1,n} - \psi_{m,n-1}), \\ f_{m-1,n}(\psi_{m-1,n+1} - \psi_{m,n}) &= i(\psi_{m,n+1} - \psi_{m-1,n}). \end{aligned} \quad (24)$$

Adding these four equations up, one obtains the 5-point scheme (22).

We are therefore lead to the following change of variables in \mathbb{Z}_e^2 (see figure 2):

$$\mu = \frac{m-n}{2}, \quad \nu = \frac{n+m}{2}, \quad (25)$$

corresponding to a $\pi/4$ rotation of the axes and, in the new variables, equation (22) is the affine self-adjoint five point scheme (11). \square

Remark. Notice that the potential ρ satisfies the constrained equation (12)

Remark. Any self-adjoint 5-point scheme (2) is gauge-equivalent to an affine self-adjoint 5-point scheme (11). Indeed, if $\sigma_{\mu,\nu}$ is a solution of (2) with coefficients $a_{\mu,\nu}$, $b_{\mu,\nu}$ and $c_{\mu,\nu}$, then

$$\psi_{\mu,\nu} = \Psi_{\mu,\nu} / \sigma_{\mu,\nu}$$

satisfies the affine equation (11) with coefficients

$$a_{\mu,\nu}^{\text{aff}} = a_{\mu,\nu} \sigma_{\mu+1,\nu} \sigma_{\mu,\nu}, \quad b_{\mu,\nu}^{\text{aff}} = b_{\mu,\nu} \sigma_{\mu,\nu+1} \sigma_{\mu,\nu}.$$

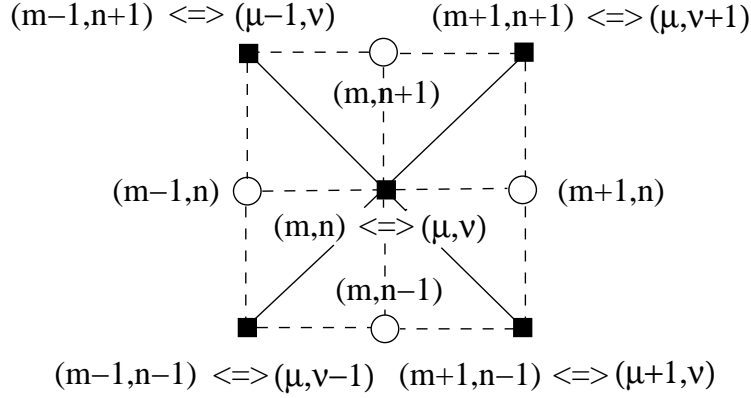


Figure 2: The white (odd) points are eliminated taking a suitable linear combination of four adjacent 4-point schemes. What remains is a $\pi/4$ rotated 5-point scheme on the black (even) points.

2.2 From the sub-lattice to the lattice

The results of the previous section 2.1 leave open the question if all the solutions $\psi : \mathbb{Z}_e^2 \rightarrow \mathbb{C}$ of the 5-point scheme (22) can be extended to solutions of the 4-point scheme (20) on the whole lattice. The answer is affirmative and the construction is very simple.

Suppose one knows a solution $\psi : \mathbb{Z}_e^2 \rightarrow \mathbb{C}$ of the 5-point scheme (22) for some given coefficients $a, b : \mathbb{Z}_e^2 \rightarrow \mathbb{R}$. Define the function $f : \mathbb{Z}^2 \rightarrow \mathbb{R}$ on the whole lattice by the inverses of equations (23) and (25), and choose the value of ψ at one arbitrary odd point of the lattice. Then, using the 4-point scheme (20) and the known values of ψ at the even points, one can construct uniquely a solution ψ of the 4-point scheme (20) whose restriction to \mathbb{Z}_e^2 coincides with ψ .

Summarizing the content of the last two sections, we have shown that any solution of the 4-point scheme (1), once restricted to \mathbb{Z}_e^2 , generates a solution of the 5-point scheme (22). Viceversa, **any** solution of the 5-point scheme (22) can be obtained restricting to \mathbb{Z}_e^2 a suitable solution of the 4-point scheme (20).

2.3 Discretization of the Schrödinger equation via the Moutard transformation

In this section we show that also the method of "discretization via transformations" [19], when applied to the elliptic Moutard (Schrödinger) equation (8), leads to the 5-point scheme thanks to the sublattice approach. Consider the general solution Ψ of the Schrödinger equation (8) together with its particular solutions θ^1 and θ^2 . One can easily check that the functions $\Psi^{(j)}$, $j = 1, 2$, given as solutions of the compatible linear systems

$$(\theta^j \Psi^{(j)})_{,x} = -\theta^j \Psi_{,y} + \theta_{,y}^j \Psi, \quad (26)$$

$$(\theta^j \Psi^{(j)})_{,y} = \theta^j \Psi_{,x} - \theta_{,x}^j \Psi, \quad (27)$$

satisfy again the Schrödinger equations (8), but with the new potentials

$$F^{(j)} = \theta^j \left[\left(\frac{1}{\theta^j} \right)_{,xx} + \left(\frac{1}{\theta^j} \right)_{,yy} \right]. \quad (28)$$

Denote by σ a solution of the system

$$\sigma_{,x} = -\theta^1 \theta^2_{,y} + \theta^1_{,y} \theta^2, \quad (29)$$

$$\sigma_{,y} = \theta^1 \theta^2_{,x} - \theta^1_{,x} \theta^2, \quad (30)$$

then the functions $\theta^{1(2)}$ and $\theta^{2(1)}$ given by

$$\sigma = \theta^2 \theta^{1(2)} = -\theta^1 \theta^{2(1)}, \quad (31)$$

satisfy the same equations as, respectively, $\Psi^{(2)}$ and $\Psi^{(1)}$. Finally, the function $\Psi^{(12)}$, obtained from the superposition formula

$$\Psi^{(12)} + \Psi = \frac{\theta^1 \theta^2}{\sigma} (\Psi^{(2)} - \Psi^{(1)}), \quad (32)$$

satisfies the Schrödinger equation (8) with potential

$$F^{(12)} = F - \frac{2}{\sigma} (\theta^1_{,x} \theta^2_{,y} - \theta^1_{,y} \theta^2_{,x}) + \sigma \left[\left(\frac{1}{\sigma} \right)_{,xx} + \left(\frac{1}{\sigma} \right)_{,yy} \right], \quad (33)$$

and is simultaneously the transform of $\Psi^{(1)}$ via $\theta^{2(1)}$, and the transform of $\Psi^{(2)}$ via $\theta^{1(2)}$.

Therefore the superposition principle of the Schrödinger equation (8) is described by the 4-point scheme (32) of the discrete Moutard type, which we know is not a proper discretization of (8). This seems to be in contradiction with the general rule that the superposition principle of a continuous system provides an integrable discretization of it. However, as we know, the sub-lattice approach resolves this problem, since the 4-point scheme (32) reduces, on its subgrid, to the 5-point scheme (see also section 2.4), which turns out to be the proper discrete analogue of (8) [24].

2.4 Different gauge forms of the 5-point scheme

For the construction of the Darboux transformations and in other applications it is useful to introduce the τ -function of the 4-point scheme (20)

$$\frac{\tau_{m+1,n} \tau_{m,n+1}}{\tau_{m,n} \tau_{m+1,n+1}} = f_{m,n}. \quad (34)$$

The fact presented below, which can be checked by direct calculation, explains the introduction of such a potential in a broader context.

Theorem 3. *Consider a lattice $\psi : \mathbb{Z}^N \rightarrow V$, $\dim V \geq N \geq 3$, satisfying the following set of linear problems*

$$\psi_{m..(i+1)..(j+1)..n} - \psi_{m..i..j..n} = g_{m..i..j..n}^{ij} (\psi_{m..(i+1)..j..n} - \psi_{m..i..(j+1)..n}), \quad i < j, \quad (35)$$

then the functions $g_{m..i..j..n}^{ij}$ can be parametrized by the potential $\tau_{m..i..j..n}$

$$g_{m..i..j..n}^{ij} = \frac{\tau_{m..(i+1)..j..n} \tau_{m..i..(j+1)..n}}{\tau_{m..(i+1)..(j+1)..n} \tau_{m..i..j..n}}, \quad i < j,$$

and the compatibility condition of the linear system (35) gives the nonlinear system of discrete BKP equations [22]

$$\begin{aligned} & \tau_{m..i..j..k..n} \tau_{m..(i+1)..(j+1)..(k+1)..n} - \tau_{m..(i+1)..j..k..n} \tau_{m..i..(j+1)..(k+1)..n} + \\ & \tau_{m..i..(j+1)..k..n} \tau_{m..(i+1)..j..(k+1)..n} - \tau_{m..i..j..(k+1)..n} \tau_{m..(i+1)..(j+1)..k..n} = 0, \quad i < j < k. \end{aligned}$$

Let us define the function

$$\Phi_{m,n} = \frac{\tau_{m,n}}{\tau_{m+1,n}} \psi_{m,n}, \quad (36)$$

then, by direct calculation using equations (22) and (23), one can check that the function Φ , restricted to the even grid \mathbb{Z}_e^2 , satisfies the equation

$$h_{\mu+1,\nu} \Phi_{\mu+1,\nu} + h_{\mu,\nu} \Phi_{\mu-1,\nu} + h_{\mu,\nu+1} \Phi_{\mu,\nu+1} + h_{\mu,\nu} \Phi_{\mu,\nu-1} = c_{\mu,\nu} \Phi_{\mu,\nu}, \quad (37)$$

where

$$h_{m,n} = \frac{\tau_{m+1,n}}{\tau_{m-1,n}}, \quad (38)$$

and

$$c_{m,n} = \frac{\tau_{m+1,n}}{\tau_{m,n}} \left(\frac{\tau_{m+1,n-1}}{\tau_{m,n-1}} + \frac{\tau_{m+1,n+1}}{\tau_{m,n+1}} \right) + \frac{\tau_{m+1,n}^2}{\tau_{m,n} \tau_{m-1,n}} \left(\frac{\tau_{m-1,n+1}}{\tau_{m,n+1}} + \frac{\tau_{m-1,n-1}}{\tau_{m,n-1}} \right). \quad (39)$$

We call equation (37) the equal-field self-adjoint 5-point scheme.

If we define the function

$$\Psi_{m,n} = \frac{\tau_{m,n}}{\sqrt{\tau_{m+1,n} \tau_{m-1,n}}} \psi_{m,n}, \quad (40)$$

then the function Ψ , restricted to the even grid \mathbb{Z}_e^2 , satisfies the discrete Schrödinger equation [24]

$$\frac{\Gamma_{\mu,\nu}}{\Gamma_{\mu+1,\nu}} \Psi_{\mu+1,\nu} + \frac{\Gamma_{\mu-1,\nu}}{\Gamma_{\mu,\nu}} \Psi_{\mu-1,\nu} + \frac{\Gamma_{\mu,\nu}}{\Gamma_{\mu,\nu+1}} \Psi_{\mu,\nu+1} + \frac{\Gamma_{\mu,\nu-1}}{\Gamma_{\mu,\nu}} \Psi_{\mu,\nu-1} = F_{\mu,\nu} \Psi_{\mu,\nu}, \quad (41)$$

where

$$\Gamma_{m,n} = \sqrt{\frac{\tau_{m-1,n}}{\tau_{m+1,n}}}, \quad (42)$$

and

$$F_{m,n} = \frac{\tau_{m-1,n}}{\tau_{m,n}} \left(\frac{\tau_{m+1,n-1}}{\tau_{m,n-1}} + \frac{\tau_{m+1,n+1}}{\tau_{m,n+1}} \right) + \frac{\tau_{m+1,n}}{\tau_{m,n}} \left(\frac{\tau_{m-1,n+1}}{\tau_{m,n+1}} + \frac{\tau_{m-1,n-1}}{\tau_{m,n-1}} \right). \quad (43)$$

Notice the following connection formulas between the equal-field and the Schrödinger gauges:

$$\Phi_{\mu,\nu} = \Gamma_{\mu,\nu} \Psi_{\mu,\nu}, \quad h_{\mu,\nu} = \frac{1}{\Gamma_{\mu,\nu}^2}, \quad c_{\mu,\nu} = \frac{1}{\Gamma_{\mu,\nu}^2} F_{\mu,\nu}. \quad (44)$$

Remark. The potential τ is defined up to multiplication by functions of single variables m and n , which implies that also Φ and Ψ are not unique.

Finally we mention a direct consequence of the above definitions of the equal-field and the Schrödinger gauges.

Corollary 4. *Given a solution $\theta : \mathbb{Z}^2 \rightarrow \mathbb{C}$ of the discrete Moutard equation (20) with the τ -function $\tau_{m,n}$, then*

a) θ restricted to the sub-lattice satisfies the affine 5-point scheme (11);

b) the function

$$\rho_{m,n} = \frac{\tau_{m,n}}{\tau_{m+1,n}} \theta_{m,n}, \quad (45)$$

restricted to the sub-lattice, satisfies the corresponding equal-field 5-point scheme (37);

c) the function

$$\Theta_{m,n} = \frac{\tau_{m,n}}{\sqrt{\tau_{m+1,n}\tau_{m-1,n}}} \theta_{m,n} = \frac{1}{\Gamma_{m,n}} \rho_{m,n}, \quad (46)$$

restricted to the sub-lattice, satisfies the corresponding Schrödinger 5-point scheme (41).

3 Darboux transformations of the sub-lattice

An obvious application of the above construction is that, once a sub-lattice of a given integrable lattice is identified, one obtains essentially for free some of its integrability properties. Here we show, for instance, the construction of the Darboux transformations [24] of the 5-point scheme in the affine (11), equal-field (37) and Schrödinger (41) forms, induced by the transformations of the 4-point lattice (20).

3.1 DTs of the affine 5-point scheme

We first recall relevant material [27] on the Darboux-Moutard transformations of the 4-point lattice.

Proposition 5. *Given a solution θ of the discrete 4-point scheme (20), then any solution $\tilde{\psi}$ of the compatible linear system of the first order*

$$\tilde{\psi}_{m+1,n} + \psi_{m,n} = \frac{\theta_{m,n}}{\theta_{m+1,n}} (\psi_{m+1,n} + \tilde{\psi}_{m,n}), \quad (47)$$

$$\tilde{\psi}_{m,n+1} + \psi_{m,n} = \frac{\theta_{m,n}}{\theta_{m,n+1}} (\psi_{m,n+1} + \tilde{\psi}_{m,n}), \quad (48)$$

satisfies the 4-point scheme (20) with the transformed potential

$$\tilde{f}_{m,n} = \frac{\theta_{m+1,n}\theta_{m,n+1}}{\theta_{m,n}\theta_{m+1,n+1}} f_{m,n}, \quad (49)$$

while the corresponding transformation of the τ -function takes the simple form

$$\tilde{\tau}_{m,n} = \theta_{m,n} \tau_{m,n}. \quad (50)$$

Remark. Notice that the function

$$\tilde{\theta}_{m,n} = \frac{1}{\theta_{m,n}} \quad (51)$$

is a solution the 4-point scheme (20) of $\tilde{\psi}$.

Corollary 6. *One can interpret the transformation as a shift in the third dimension of the lattice, and the transformation equations (47)-(48) as linear problems of the form (20) involving that dimension. In particular, the transformation rule of the τ -function (50) and the parametrization (34) of the potential $f_{m,n}$ in equation (20) satisfied by $\theta_{m,n}$ lead to equation*

$$\tau_{m,n}\tilde{\tau}_{m+1,n+1} - \tilde{\tau}_{m,n}\tau_{m+1,n+1} = i(\tau_{m,n+1}\tilde{\tau}_{m+1,n} - \tau_{m+1,n}\tilde{\tau}_{m,n+1}), \quad (52)$$

of the form of the discrete BKP equation for $N = 3$.

Again, the function $\tilde{\psi}_{m,n}$, when restricted to the even (or odd) lattice, satisfies the affine 5-point scheme (11). However, the transformation equations (47)-(48) depend on values of the transformation potential θ on the full lattice. Our goal is to constrain the DT of the 5-point scheme to the sub-lattice as well. Obviously, given a solution of the affine 5-point scheme on the sub-lattice, it can be propagated to the full lattice in the spirit of section 2.2 and then used to construct the transformation. We will show, however, that the transformations of the 5-point scheme can be done, in a more elegant way. In particular we will constrain the transformation equations to the sub-lattice.

Lemma 7. *Let $\psi_{m,n}$ and $\theta_{m,n}$ be solutions of (20), and let $\tilde{\psi}_{m,n}$ be the transformed solution constructed via (47)-(48). Then the function $\hat{\psi}_{m,n}$, defined by*

$$\hat{\psi}_{m,n} = i\theta_{m-1,n}\tilde{\psi}_{m-1,n}, \quad (53)$$

with the functions $a_{m,n}$ and $b_{m,n}$ defined by equations (23), satisfies

$$\hat{\psi}_{m+1,n-1} - \hat{\psi}_{m,n} = b_{m-1,n-1}(\theta_{m-1,n+1}\psi_{m,n} - \theta_{m,n}\psi_{m-1,n+1}), \quad (54)$$

$$\hat{\psi}_{m+1,n+1} - \hat{\psi}_{m,n} = -a_{m-1,n+1}(\theta_{m-1,n+1}\psi_{m,n} - \theta_{m,n}\psi_{m-1,n+1}). \quad (55)$$

Proof. Subtract equation (48) from equation (47), use the four point scheme (20) and evaluate the result at $(m-1, n-1)$ to get equation (54). Similarly, add equation (47) evaluated at $(m-1, n)$ to equation (48) to obtain equation (55). \square

Corollary 8. *The function $\hat{\psi}_{m,n}$ satisfies equation*

$$\begin{aligned} \hat{a}_{m,n}(\hat{\psi}_{m+1,n-1} - \hat{\psi}_{m,n}) + \hat{a}_{m-1,n+1}(\hat{\psi}_{m-1,n+1} - \hat{\psi}_{m,n}) + \\ \hat{b}_{m,n}(\hat{\psi}_{m+1,n+1} - \hat{\psi}_{m,n}) + \hat{b}_{m-1,n-1}(\hat{\psi}_{m-1,n-1} - \hat{\psi}_{m,n}) = 0, \end{aligned} \quad (56)$$

with coefficients

$$\hat{a}_{m,n} = \frac{1}{b_{m-1,n-1}\theta_{m,n}\theta_{m-1,n-1}}, \quad \hat{b}_{m,n} = \frac{1}{a_{m-1,n+1}\theta_{m,n}\theta_{m-1,n+1}}. \quad (57)$$

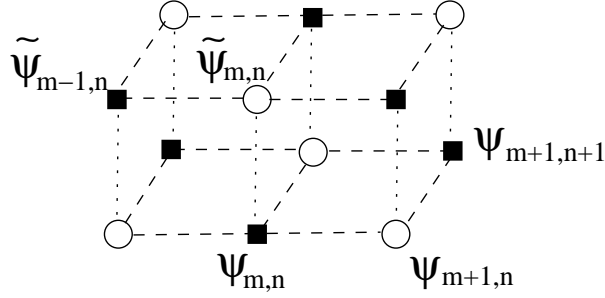


Figure 3: The sub-lattice reduction of the discrete Moutard transformation.

Proof. The function $\tilde{\psi}_{m,n}$ satisfies equation (22) with coefficients $\tilde{a}_{m,n}$ and $\tilde{b}_{m,n}$ obtained from $\tilde{f}_{m,n}$ via equations (23). Because the function $1/\theta_{m,n}$ satisfies the same equation as $\tilde{\psi}_{m,n}$, then $\theta_{m,n}\tilde{\psi}_{m,n}$ satisfies again equation (22), but with new coefficients (compare with remark 2 after proposition 2). \square

The following theorem can be obtained just restricting the above results to the sub-lattice, but it can be also proven directly.

Theorem 9. *Let $\psi_{\mu,\nu}$ satisfy the self-adjoint affine 5-point scheme (11) and let $\theta_{\mu,\nu}$ be a particular solution of (11). Then the solution $\hat{\psi}_{\mu,\nu}$ of the compatible linear system*

$$\hat{\psi}_{\mu+1,\nu} - \hat{\psi}_{\mu,\nu} = b_{\mu,\nu-1}(\theta_{\mu,\nu-1}\psi_{\mu,\nu} - \theta_{\mu,\nu}\psi_{\mu,\nu-1}), \quad (58)$$

$$\hat{\psi}_{\mu,\nu+1} - \hat{\psi}_{\mu,\nu} = -a_{\mu-1,\nu}(\theta_{\mu-1,\nu}\psi_{\mu,\nu} - \theta_{\mu,\nu}\psi_{\mu-1,\nu}), \quad (59)$$

satisfies the self-adjoint affine 5-point scheme (11) with transformed coefficients

$$\hat{a}_{\mu,\nu} = \frac{1}{b_{\mu,\nu-1}\theta_{\mu,\nu}\theta_{\mu,\nu-1}}, \quad \hat{b}_{\mu,\nu} = \frac{1}{a_{\mu-1,\nu}\theta_{\mu,\nu}\theta_{\mu-1,\nu}}. \quad (60)$$

Proof. The compatibility condition of (58)-(59) is the self-adjoint affine 5-point scheme (11). Equations (58)-(59) and their compatibility condition imply the self-adjoint affine 5-point scheme (11) for $\hat{\psi}_{\mu,\nu}$ with coefficients given by (60). \square

Remark. The transformation (58)-(59) is a restriction of the DT of the general self-adjoint 5-point scheme given in [24] to the class of self-adjoint affine 5-point schemes.

Remark. Equations (58)-(59) are obtained restricting equations (54)-(55) to the sub-lattice, while equation (60) is the corresponding restriction of equations (57).

It seems that the transformation acts from the even (black) lattice \mathbb{Z}_e^2 to the odd (white) lattice \mathbb{Z}_o^2 (or vice versa). However, the proper point of view is to consider (see corollary 6) the transformation as the shift into an additional dimension of the lattice, and then the transformation acts between black (or between white) points of the lattice (see figure 3). There are three other equivalent choices of the transformation on the small lattice which give also restrictions of the DT to the sub-lattice. The resulting transformation formulas differ from (58)-(59) in some shifts. The present form has been chosen to be in agreement with the earlier formulas of [24].

3.2 DTs of the self-adjoint equal field 5-point scheme

Let us rewrite equations (54)-(55) using, instead of the fields $\psi_{m,n}$, $a_{m,n}$, $b_{m,n}$ and $\theta_{m,n}$, the fields $\Phi_{m,n}$, $h_{m,n}$ and $\rho_{m,n}$ defined by equations (36), (38) and (45). The natural counterpart of the function $\hat{\psi}_{m,n}$ given by equation (53) is the function

$$\hat{\Phi}_{m,n} = i\tilde{\Phi}_{m-1,n}, \quad (61)$$

where $\tilde{\Phi}_{m,n}$ is given by the transformed version of equation (36), i.e.,

$$\tilde{\Phi}_{m,n} = \frac{\tilde{\tau}_{m,n}}{\tilde{\tau}_{m+1,n}} \tilde{\psi}_{m,n} = \frac{\theta_{m,n} \tilde{\psi}_{m,n}}{\rho_{m+1,n} h_{m+1,n}}. \quad (62)$$

The counterpart of the DT of the affine self-adjoint 5-point scheme described in theorem 9 will be the following theorem on the DT of the self-adjoint equal field 5-point scheme.

Theorem 10. *Let $\Phi_{\mu,\nu}$ and $\rho_{\mu,\nu}$ be solutions of the self-adjoint equal field 5-point scheme (37), then the solution $\hat{\Phi}_{\mu,\nu}$ of the following linear system of the first order*

$$\rho_{\mu+1,\nu} h_{\mu+1,\nu} \hat{\Phi}_{\mu+1,\nu} - \rho_{\mu,\nu} h_{\mu,\nu} \hat{\Phi}_{\mu,\nu} = h_{\mu,\nu} (\rho_{\mu,\nu-1} \Phi_{\mu,\nu} - \rho_{\mu,\nu} \Phi_{\mu,\nu-1}), \quad (63)$$

$$\rho_{\mu,\nu+1} h_{\mu,\nu+1} \hat{\Phi}_{\mu,\nu+1} - \rho_{\mu,\nu} h_{\mu,\nu} \hat{\Phi}_{\mu,\nu} = -h_{\mu,\nu} (\rho_{\mu-1,\nu} \Phi_{\mu,\nu} - \rho_{\mu,\nu} \Phi_{\mu-1,\nu}), \quad (64)$$

satisfies the self-adjoint equal field 5-point scheme with coefficients

$$\hat{h}_{\mu,\nu} = \frac{h_{\mu,\nu} \rho_{\mu,\nu}}{\rho_{\mu-1,\nu-1}}, \quad (65)$$

and

$$\hat{c}_{\mu,\nu} = h_{\mu,\nu} \rho_{\mu,\nu} \left(\frac{1}{\rho_{\mu,\nu-1}} + \frac{1}{\rho_{\mu-1,\nu}} \right) + \frac{(h_{\mu,\nu} \rho_{\mu,\nu})^2}{\rho_{\mu-1,\nu-1}} \left(\frac{1}{h_{\mu-1,\nu} \rho_{\mu-1,\nu}} + \frac{1}{h_{\mu,\nu-1} \rho_{\mu,\nu-1}} \right). \quad (66)$$

Proof. The theorem can be verified by direct calculation. To do its derivation within the sub-lattice theory (the form of the transformation equations has been explained before the formulation of the theorem), notice that the function $\hat{h}_{\mu,\nu}$ is restriction of

$$\hat{h}_{m,n} = \tilde{h}_{m-1,n}, \quad \text{with} \quad \tilde{h}_{m,n} = \frac{\tilde{\tau}_{m+1,n}}{\tilde{\tau}_{m-1,n}}. \quad (67)$$

Also the transformed coefficient $\hat{c}_{\mu,\nu}$ is the restriction of

$$\hat{c}_{m,n} = \tilde{c}_{m-1,n}, \quad (68)$$

with $\tilde{c}_{m,n}$ given by equation (39), with $\tilde{\tau}_{m,n}$ instead of $\tau_{m,n}$. In putting it on the sub-lattice one uses equation (20) satisfied by $\theta_{m,n}$. \square

Remark. In terms of the function

$$\hat{\rho}_{\mu,\nu} = \frac{1}{\rho_{\mu,\nu} h_{\mu,\nu}}, \quad (69)$$

formula (66) takes the form

$$\hat{c}_{\mu,\nu} = \frac{1}{\hat{\rho}_{\mu,\nu}} \left(\hat{h}_{\mu+1,\nu} \hat{\rho}_{\mu+1,\nu} + \hat{h}_{\mu,\nu} \hat{\rho}_{\mu-1,\nu} + \hat{h}_{\mu,\nu+1} \hat{\rho}_{\mu,\nu+1} + \hat{h}_{\mu,\nu} \hat{\rho}_{\mu,\nu-1} \right), \quad (70)$$

i.e., the function $\hat{\rho}_{\mu,\nu}$ is a solution of the equation satisfied by $\hat{\Phi}_{m,n}$. This is the 5-point analogue of the fact that the function $\tilde{\theta}_{m,n} = 1/\theta_{m,n}$ satisfies the transformed equation (20). Indeed, the function $\hat{\rho}_{\mu,\nu}$ is the restriction

$$\hat{\rho}_{m,n} = \tilde{\rho}_{m-1,n} \quad (71)$$

of the function

$$\tilde{\rho}_{m,n} = \frac{\tilde{\tau}_{m,n}}{\tilde{\tau}_{m+1,n}} \tilde{\theta}_{m,n}, \quad (72)$$

obtained from $\tilde{\theta}_{m,n}$ in the same way as $\hat{\Phi}_{m,n}$ is obtained from $\tilde{\psi}_{m,n}$.

3.3 DTs of the Schrödinger 5-point scheme

In what follows we will obtain the DT of the 5-point Schrödinger scheme [24] from the sub-lattice theory. In order not to repeat the full procedure of the previous two subsections, we will use the connection formulas (44) between the equal-field and Schrödinger gauges and the form (46) of the particular solution $\Theta_{\mu,\nu}$ of the Schrödinger scheme. The transformed potential $\hat{\Gamma}_{\mu,\nu}$ can be obtained from the connection formulas (44) and the potential $\hat{h}_{\mu,\nu}$ of the equal-field scheme given by equation (65), i.e.,

$$\hat{\Gamma}_{\mu,\nu}^2 = \frac{1}{\hat{h}_{\mu,\nu}} = \Gamma_{\mu,\nu} \Gamma_{\mu-1,\nu-1} \frac{\Theta_{\mu-1,\nu-1}}{\Theta_{\mu,\nu}}. \quad (73)$$

The transformed solution $\hat{\Psi}_{\mu,\nu}$, because of (44), is related to $\hat{\Phi}_{\mu,\nu}$ by

$$\hat{\Psi}_{\mu,\nu} = \frac{\hat{\Phi}_{\mu,\nu}}{\hat{\Gamma}_{\mu,\nu}}. \quad (74)$$

The transformed potential $\hat{F}_{\mu,\nu}$ can also be obtained from the connection formulas (44)

$$\hat{F}_{\mu,\nu} = \hat{\Gamma}_{\mu,\nu}^2 \hat{c}_{\mu,\nu} = \frac{\Gamma_{\mu-1,\nu-1} \Theta_{\mu-1,\nu-1}}{\Gamma_{\mu,\nu-1} \Theta_{\mu,\nu-1}} + \frac{\Gamma_{\mu-1,\nu-1} \Theta_{\mu-1,\nu-1}}{\Gamma_{\mu-1,\nu} \Theta_{\mu-1,\nu}} + \frac{\Theta_{\mu,\nu}}{\Gamma_{\mu,\nu}} \left(\frac{\Gamma_{\mu-1,\nu}}{\Theta_{\mu-1,\nu}} + \frac{\Gamma_{\mu,\nu-1}}{\Theta_{\mu,\nu-1}} \right). \quad (75)$$

Rewriting theorem 10 in the new fields, we obtain the DT of the Schrödinger 5-point scheme.

Theorem 11. *Let $\Psi_{\mu,\nu}$ and $\Theta_{\mu,\nu}$ be solutions of the Schrödinger 5-point scheme (41); then the solution $\hat{\Psi}_{\mu,\nu}$ of the following linear system of the first order*

$$\frac{\hat{\Gamma}_{\mu+1,\nu}}{\Gamma_{\mu+1,\nu}} \Theta_{\mu+1,\nu} \hat{\Psi}_{\mu+1,\nu} - \frac{\hat{\Gamma}_{\mu,\nu}}{\Gamma_{\mu,\nu}} \Theta_{\mu,\nu} \hat{\Psi}_{\mu,\nu} = \frac{\Gamma_{\mu,\nu-1}}{\Gamma_{\mu,\nu}} (\Theta_{\mu,\nu-1} \Psi_{\mu,\nu} - \Theta_{\mu,\nu} \Psi_{\mu,\nu-1}), \quad (76)$$

$$\frac{\hat{\Gamma}_{\mu,\nu+1}}{\Gamma_{\mu,\nu+1}} \Theta_{\mu,\nu+1} \hat{\Psi}_{\mu,\nu+1} - \frac{\hat{\Gamma}_{\mu,\nu}}{\Gamma_{\mu,\nu}} \Theta_{\mu,\nu} \hat{\Psi}_{\mu,\nu} = -\frac{\Gamma_{\mu-1,\nu}}{\Gamma_{\mu,\nu}} (\Theta_{\mu-1,\nu} \Psi_{\mu,\nu} - \Theta_{\mu,\nu} \Psi_{\mu-1,\nu}), \quad (77)$$

where $\hat{\Gamma}_{\mu,\nu}$ is given by equation (73), satisfies the 5-point Schrödinger scheme with the coefficients $\hat{\Gamma}_{\mu,\nu}$ and the coefficients $\hat{F}_{\mu,\nu}$ given by (75).

Remark. The analogue of the self-transformed solution $\hat{\rho}_{\mu,\nu}$, given by equation (69), of the equal-field scheme is the solution

$$\hat{\Theta}_{\mu,\nu} = \frac{\hat{\rho}_{\mu,\nu}}{\hat{\Gamma}_{\mu,\nu}} = \frac{\Gamma_{\mu,\nu}}{\hat{\Gamma}_{\mu,\nu}\Theta_{\mu,\nu}} \quad (78)$$

of the transformed Schrödinger scheme. In terms of $\hat{\Theta}_{\mu,\nu}$ the formula (75) for the transformed potential $\hat{F}_{\mu,\nu}$ takes the natural form

$$\hat{F}_{\mu,\nu} = \frac{1}{\hat{\Theta}_{\mu,\nu}} \left(\frac{\hat{\Gamma}_{\mu,\nu}}{\hat{\Gamma}_{\mu+1,\nu}} \hat{\Theta}_{\mu+1,\nu} + \frac{\hat{\Gamma}_{\mu-1,\nu}}{\hat{\Gamma}_{\mu,\nu}} \hat{\Theta}_{\mu-1,\nu} + \frac{\hat{\Gamma}_{\mu,\nu}}{\hat{\Gamma}_{\mu,\nu+1}} \hat{\Theta}_{\mu,\nu+1} + \frac{\hat{\Gamma}_{\mu,\nu-1}}{\hat{\Gamma}_{\mu,\nu}} \hat{\Theta}_{\mu,\nu-1} \right). \quad (79)$$

4 Superposition of the DTs

Using the superposition principle of the DTs of the discrete Moutard equation [27], we will obtain the corresponding superposition principles for the DTs of the self-adjoint 5-point schemes in the equal field gauge and in the Schrödinger gauge.

Proposition 12. *Given solutions $\psi_{m,n}$, $\theta_{m,n}^1$ and $\theta_{m,n}^2$ of the 4-point scheme (20), denote by $\psi_{m,n}^{(1)}$ and $\psi_{m,n}^{(2)}$ the transforms respectively of $\psi_{m,n}$ via $\theta_{m,n}^1$ and $\theta_{m,n}^2$. If $\sigma_{m,n}$ is a solution of the compatible linear system*

$$\sigma_{m+1,n} - \sigma_{m,n} = \theta_{m,n}^2 \theta_{m+1,n}^1 - \theta_{m,n}^1 \theta_{m+1,n}^2, \quad (80)$$

$$\sigma_{m,n+1} - \sigma_{m,n} = \theta_{m,n}^2 \theta_{m,n+1}^1 - \theta_{m,n}^1 \theta_{m,n+1}^2, \quad (81)$$

then $\theta_{m,n}^{1(2)}$ and $\theta_{m,n}^{2(1)}$, given by

$$\theta_{m,n}^{1(2)} \theta_{m,n}^2 = -\theta_{m,n}^{2(1)} \theta_{m,n}^1 = \sigma_{m,n}, \quad (82)$$

are solutions of the 4-points schemes satisfied respectively by $\psi_{m,n}^{(2)}$ and $\psi_{m,n}^{(1)}$. Moreover, the function $\psi_{m,n}^{(12)}$ given by

$$\psi_{m,n}^{(12)} - \psi_{m,n} = \frac{\theta_{m,n}^1 \theta_{m,n}^2}{\sigma_{m,n}} (\psi_{m,n}^{(1)} - \psi_{m,n}^{(2)}), \quad (83)$$

is simultaneously the transform of $\psi_{m,n}^{(1)}$ via $\theta_{m,n}^{2(1)}$ and the transform of $\psi_{m,n}^{(2)}$ via $\theta_{m,n}^{1(2)}$. The corresponding transformation of the τ -function is given by

$$\tau_{m,n}^{(12)} = \sigma_{m,n} \tau_{m,n}. \quad (84)$$

Our goal will be to derive the corresponding Bianchi superposition principle for the 5-point equal-field scheme, and then for the Schrödinger scheme. First, we will rewrite equations (80)-(81) in terms of the transformation data

$$\rho_{m,n}^j = \frac{\tau_{m,n}}{\tau_{m+1,n}} \theta_{m,n}^j, \quad j = 1, 2, \quad (85)$$

of the equal-field gauge (see equation(45)), and we will put them on the sub-lattice.

From equations (80)-(81) and the discrete Moutard equation (20) we have

$$\sigma_{m,n-1} - \sigma_{m-1,n} = -i \frac{\tau_{m,n} \tau_{m-1,n-1}}{\tau_{m-1,n} \tau_{m,n-1}} (\theta_{m,n}^1 \theta_{m-1,n-1}^2 - \theta_{m,n}^2 \theta_{m-1,n-1}^1), \quad (86)$$

$$\sigma_{m,n+1} - \sigma_{m-1,n} = i \frac{\tau_{m,n} \tau_{m-1,n+1}}{\tau_{m-1,n} \tau_{m,n+1}} (\theta_{m,n}^1 \theta_{m-1,n+1}^2 - \theta_{m,n}^2 \theta_{m-1,n+1}^1). \quad (87)$$

If we introduce the function

$$\Sigma_{m,n} = i \sigma_{m-1,n} \quad (88)$$

then the above equations can be rewritten in the form

$$\Sigma_{m+1,n-1} - \Sigma_{m,n} = h_{m,n} (\rho_{m,n}^1 \rho_{m-1,n-1}^2 - \rho_{m,n}^2 \rho_{m-1,n-1}^1), \quad (89)$$

$$\Sigma_{m+1,n+1} - \Sigma_{m,n} = -h_{m,n} (\rho_{m,n}^1 \rho_{m-1,n+1}^2 - \rho_{m,n}^2 \rho_{m-1,n+1}^1). \quad (90)$$

The functions $\Phi_{m,n}^{(j)}$, constructed according to equations (61)-(62)

$$\Phi_{m,n}^{(j)} = i \frac{\theta_{m-1,n}^j \psi_{m-1,n}^{(j)}}{\rho_{m,n}^j h_{m,n}}, \quad j = 1, 2, \quad (91)$$

are the transforms of $\Phi_{\mu,\nu}$ via $\rho_{\mu,\nu}^j$, when restricted to the sub-lattice. Consequently, the transformed data $\rho_{\mu,\nu}^{j(k)}$, $j \neq k$ are restrictions of the functions

$$\rho_{m,n}^{j(k)} = i \frac{\theta_{m-1,n}^k \theta_{m-1,n}^{j(k)}}{\rho_{m,n}^k h_{m,n}}, \quad j \neq k, \quad (92)$$

which, due to equation (82), are related to $\Sigma_{m,n}$ by

$$h_{m,n} \rho_{m,n}^2 \rho_{m,n}^{1(2)} = -h_{m,n} \rho_{m,n}^1 \rho_{m,n}^{2(1)} = \Sigma_{m,n}. \quad (93)$$

Using equations (47)-(48) we modify the superposition formula (83) in order to include functions $\psi_{m-1,n}^{(1)}$ and $\psi_{m-1,n}^{(2)}$ instead of $\psi_{m,n}^{(1)}$ and $\psi_{m,n}^{(2)}$

$$\sigma_{m,n} \psi_{m,n}^{(12)} - \sigma_{m-1,n} \psi_{m,n} = \theta_{m,n}^2 \theta_{m-1,n}^1 \psi_{m-1,n}^{(1)} - \theta_{m,n}^1 \theta_{m-1,n}^2 \psi_{m-1,n}^{(2)}. \quad (94)$$

Finally, applying twice equations (61)-(62), we obtain that the function

$$\Phi_{m,n}^{(12)} = - \frac{\tau_{m-2,n}^{(12)}}{\tau_{m-1,n}^{(12)}} \psi_{m-2,n}^{(12)}, \quad (95)$$

gives the desired superposition $\Phi_{\mu,\nu}^{(12)}$, when restricted to the sub-lattice.

Putting all the above considerations together, we obtain the Bianchi-type permutability theorem, which can also be verified by direct calculation.

Theorem 13. Given solutions $\Phi_{\mu,\nu}$, $\rho_{\mu,\nu}^1$ and $\rho_{\mu,\nu}^2$ of the 5-point equal field scheme (37), denote by $\Phi_{\mu,\nu}^{(1)}$ and $\Phi_{\mu,\nu}^{(2)}$ the transforms respectively of $\Phi_{\mu,\nu}$ via $\rho_{\mu,\nu}^1$ and $\rho_{\mu,\nu}^2$. If $\Sigma_{\mu,\nu}$ is a solution of the compatible linear system

$$\Sigma_{\mu+1,\nu} - \Sigma_{\mu,\nu} = h_{\mu,\nu}(\rho_{\mu,\nu}^1 \rho_{\mu,\nu-1}^2 - \rho_{\mu,\nu}^2 \rho_{\mu,\nu-1}^1), \quad (96)$$

$$\Sigma_{\mu,\nu+1} - \Sigma_{\mu,\nu} = -h_{\mu,\nu}(\rho_{\mu,\nu}^1 \rho_{\mu-1,\nu}^2 - \rho_{\mu,\nu}^2 \rho_{\mu-1,\nu}^1), \quad (97)$$

then the functions $\rho_{\mu,\nu}^{1(2)}$ and $\rho_{\mu,\nu}^{2(1)}$ given by

$$h_{\mu,\nu} \rho_{\mu,\nu}^{1(2)} \rho_{\mu,\nu}^2 = -h_{\mu,\nu} \rho_{\mu,\nu}^{2(1)} \rho_{\mu,\nu}^1 = \Sigma_{\mu,\nu}, \quad (98)$$

are solutions of the 5-points schemes satisfied by $\Phi_{\mu,\nu}^{(2)}$ and $\Phi_{\mu,\nu}^{(1)}$, correspondingly. Moreover the function $\Phi_{\mu,\nu}^{(12)}$, given by

$$\frac{\Sigma_{\mu+1,\nu+1}}{\Sigma_{\mu,\nu}} \Phi_{\mu+1,\nu+1}^{(12)} + \Phi_{\mu,\nu} = \frac{h_{\mu,\nu} \rho_{\mu,\nu}^1 \rho_{\mu,\nu}^2}{\Sigma_{\mu,\nu}} (\Phi_{\mu,\nu}^{(2)} - \Phi_{\mu,\nu}^{(1)}), \quad (99)$$

is simultaneously the transform of $\Phi_{\mu,\nu}^{(1)}$ via $\rho_{\mu,\nu}^{2(1)}$ and the transform of $\Phi_{\mu,\nu}^{(2)}$ via $\rho_{\mu,\nu}^{1(2)}$.

Corollary 14. Equations (65) and (66), together with the above theorem, imply the transformation law of the coefficients of the doubly transformed equal field scheme

$$h_{\mu+1,\nu+1}^{(12)} = \frac{\Sigma_{\mu+1,\nu+1}}{\Sigma_{\mu,\nu}} h_{\mu,\nu}, \quad (100)$$

$$c_{\mu+1,\nu+1}^{(12)} = \frac{\Sigma_{\mu+1,\nu+1}^2}{\Sigma_{\mu+1,\nu} \Sigma_{\mu,\nu+1}} \left(c_{\mu,\nu} + \frac{h_{\mu,\nu}^2}{\Sigma_{\mu,\nu}} (\rho_{\mu,\nu-1}^1 \rho_{\mu-1,\nu}^2 - \rho_{\mu-1,\nu}^1 \rho_{\mu,\nu-1}^2) + \right. \\ \left. + \frac{h_{\mu+1,\nu} h_{\mu,\nu+1}}{\Sigma_{\mu+1,\nu+1}} (\rho_{\mu,\nu+1}^1 \rho_{\mu+1,\nu}^2 - \rho_{\mu+1,\nu}^1 \rho_{\mu,\nu+1}^2) \right). \quad (101)$$

Finally, let us formulate the corresponding superposition principle for the Schrödinger scheme. It follows just from the connection formulas (44) and (46) applied to theorem 13.

Theorem 15. Let $\Psi_{\mu,\nu}$, $\Theta_{\mu,\nu}^1$ and $\Theta_{\mu,\nu}^2$ be solutions of the 5-point Schrödinger scheme (41), denote by $\Psi_{\mu,\nu}^{(1)}$ and $\Psi_{\mu,\nu}^{(2)}$ the transforms respectively of $\Psi_{\mu,\nu}$ via $\Theta_{\mu,\nu}^1$ and $\Theta_{\mu,\nu}^2$, and let $\Gamma_{\mu,\nu}^{(1)}$, $\Gamma_{\mu,\nu}^{(2)}$ denote the first coefficients of the corresponding equations, i.e.,

$$\left[\Gamma_{\mu,\nu}^{(j)} \right]^2 = \frac{\Theta_{\mu-1,\nu-1}^j}{\Theta_{\mu,\nu}^j} \Gamma_{\mu,\nu} \Gamma_{\mu-1,\nu-1}, \quad j = 1, 2. \quad (102)$$

If $\Sigma_{\mu,\nu}$ is a solution of the compatible linear system

$$\Sigma_{\mu+1,\nu} - \Sigma_{\mu,\nu} = \frac{\Gamma_{\mu,\nu-1}}{\Gamma_{\mu,\nu}} (\Theta_{\mu,\nu}^1 \Theta_{\mu,\nu-1}^2 - \Theta_{\mu,\nu}^2 \Theta_{\mu,\nu-1}^1), \quad (103)$$

$$\Sigma_{\mu,\nu+1} - \Sigma_{\mu,\nu} = -\frac{\Gamma_{\mu-1,\nu}}{\Gamma_{\mu,\nu}} (\Theta_{\mu,\nu}^1 \Theta_{\mu-1,\nu}^2 - \Theta_{\mu,\nu}^2 \Theta_{\mu-1,\nu}^1), \quad (104)$$

then $\Theta_{\mu,\nu}^{1(2)}$ and $\Theta_{\mu,\nu}^{2(1)}$, given by

$$\Theta_{\mu,\nu}^{1(2)} \Theta_{\mu,\nu}^{2(1)} \frac{\Gamma_{\mu,\nu}^{(2)}}{\Gamma_{\mu,\nu}} = -\Theta_{\mu,\nu}^{2(1)} \Theta_{\mu,\nu}^{1(2)} \frac{\Gamma_{\mu,\nu}^{(1)}}{\Gamma_{\mu,\nu}} = \Sigma_{\mu,\nu}, \quad (105)$$

are solutions of the 5-points schemes satisfied respectively by $\Psi_{\mu,\nu}^{(2)}$ and $\Psi_{\mu,\nu}^{(1)}$. The function $\Psi_{\mu,\nu}^{(12)}$, given by

$$\frac{\Sigma_{\mu+1,\nu+1}}{\Sigma_{\mu,\nu}} \Gamma_{\mu+1,\nu+1}^{(12)} \Psi_{\mu+1,\nu+1}^{(12)} + \Gamma_{\mu,\nu} \Psi_{\mu,\nu} = \frac{\Theta_{\mu,\nu}^1 \Theta_{\mu,\nu}^2}{\Sigma_{\mu,\nu}} (\Gamma_{\mu,\nu}^{(2)} \Psi_{\mu,\nu}^{(2)} - \Gamma_{\mu,\nu}^{(1)} \Psi_{\mu,\nu}^{(1)}), \quad (106)$$

where

$$\left[\Gamma_{\mu+1,\nu+1}^{(12)} \right]^2 = \frac{\Sigma_{\mu,\nu}}{\Sigma_{\mu+1,\nu+1}} \Gamma_{\mu,\nu}^2, \quad (107)$$

is simultaneously the transform of $\Psi_{\mu,\nu}^{(1)}$ via $\Theta_{\mu,\nu}^{2(1)}$ and the transform of $\Psi_{\mu,\nu}^{(2)}$ via $\Theta_{\mu,\nu}^{1(2)}$. It satisfies the 5-point Schrödinger scheme (41) with the coefficient $\Gamma_{\mu,\nu}^{(12)}$ given in (107), and with the coefficient $F_{\mu,\nu}^{(12)}$ given by

$$F_{\mu+1,\nu+1}^{(12)} = \frac{\Sigma_{\mu+1,\nu+1} \Sigma_{\mu,\nu}}{\Sigma_{\mu+1,\nu} \Sigma_{\mu,\nu+1}} \left(F_{\mu,\nu} + \frac{\Gamma_{\mu-1,\nu} \Gamma_{\mu,\nu-1}}{\Sigma_{\mu,\nu} \Gamma_{\mu,\nu}^2} (\Theta_{\mu,\nu-1}^1 \Theta_{\mu-1,\nu}^2 - \Theta_{\mu-1,\nu}^1 \Theta_{\mu,\nu-1}^2) + \right. \\ \left. + \frac{\Gamma_{\mu,\nu}^2}{\Sigma_{\mu+1,\nu+1} \Gamma_{\mu+1,\nu} \Gamma_{\mu,\nu+1}} (\Theta_{\mu,\nu+1}^1 \Theta_{\mu+1,\nu}^2 - \Theta_{\mu+1,\nu}^1 \Theta_{\mu,\nu+1}^2) \right). \quad (108)$$

5 Algebro - geometric solutions

In this sections we construct algebro - geometric solutions of the 4-point scheme (20) and, due to the results of the previous section, of the 5-point scheme (22),(11).

The direct and inverse periodic transform for a generic 4-point scheme (12) was developed in [18]. To obtain algebro-geometric solutions of a specific 4-point scheme we have to impose suitable constraints on the spectral data. As we shall see, the constraints which give rise to the 4-point scheme (20) are analogous to those introduced in [30] in the study of the 2D continuous Schrödinger operator. Constraints of such type were first introduced in [6] in the theory of reductions of 1+1 systems. In the theory of discrete systems, analogous reductions were used in [1] to characterize orthogonal and Egoroff quadrilateral lattices.

We remark that algebro-geometric 5-point schemes were also studied in [18]. The main difference between our 5-point schemes and the schemes studied in [18] is the following. In our case, the full Bloch variety for zero energy is constructed and the eigenfunctions with other energies are not incorporated in the spectral transform. Moreover, only self-adjoint schemes are constructed. In [18], generic schemes with the following property were studied: for each eigenvalue, a finite number of eigenfunctions are explicitly constructed.

The standard finite-gap construction [18] for a generic 4-point scheme is based on the following spectral data. Assume that we have:

1. A compact, regular, connected Riemann surface Γ of genus g .

2. $l + 1$ points R_1, \dots, R_{l+1} in Γ – the normalization points for the wave function (denoted by Ψ).
3. $l + g$ points $\gamma_1, \dots, \gamma_{l+g}$ in Γ – the divisor of poles of the wave function.
4. A collection of points $P_1^+, \dots, P_M^+, P_1^-, \dots, P_M^-, Q_1^+, \dots, Q_N^+, Q_1^-, \dots, Q_N^-$, where M, N are arbitrary positive integers.

From the Riemann-Roch theorem it follows that, for generic data, there exists an unique function $\Psi(\gamma, m, n)$, where $\gamma \in \Gamma$, $m, n \in \mathbb{Z}$, $1 \leq m \leq M$, $1 \leq n \leq N$, with the following properties:

1. $\Psi(\gamma, m, n)$ is a meromorphic function of γ in Γ .
2. $\Psi(\gamma, m, n)$ has at most first-order poles at the points γ_k , $k = 1, \dots, g + l$, P_k^+ , $k = 1, \dots, m$, Q_k^+ , $k = 1, \dots, n$ and no other singularities.
3. $\Psi(\gamma, m, n)$ has at least first-order zeroes at the points P_k^- , $k = 1, \dots, m$, Q_k^- , $k = 1, \dots, n$.
4. $\Psi(R_k, m, n) = 1$, $k = 1, \dots, l + 1$.

The discrete time shift $m \rightarrow m + 1$ corresponds to adding one extra pole P_{m+1}^+ and one extra zero P_{m+1}^- to the wave function.

Let us check that function $\Psi(\gamma, m, n)$ satisfies the 4-point equation

$$\Psi(\gamma, m+1, n+1) + \alpha_1(m, n)\Psi(\gamma, m+1, n) + \alpha_2(m, n)\Psi(\gamma, m, n+1) + \alpha_3(m, n)\Psi(\gamma, m, n) = 0, \quad (109)$$

in which the γ -independent coefficients $\alpha_1(m, n)$, $\alpha_2(m, n)$, $\alpha_3(m, n)$ are defined by the following formulas:

$$\alpha_1(m, n) = - \lim_{\gamma \rightarrow P_{m+1}^+} \frac{\Psi(\gamma, m+1, n+1)}{\Psi(\gamma, m+1, n)} \quad (110)$$

$$\alpha_2(m, n) = - \lim_{\gamma \rightarrow Q_{n+1}^+} \frac{\Psi(\gamma, m+1, n+1)}{\Psi(\gamma, m, n+1)} \quad (111)$$

$$\alpha_3(m, n) = -1 - \alpha_1(m, n) - \alpha_2(m, n). \quad (112)$$

Indeed the left-hand side of (109) has the following properties:

1. It is a meromorphic function of γ in Γ .
2. It has at most first-order poles at the points γ_k , $k = 1, \dots, g + l$, P_k^+ , $k = 1, \dots, m$, Q_k^+ , $k = 1, \dots, n$ and no other singularities. (Conditions (110), (111) mean exactly that the poles at the points P_{m+1}^+ , Q_{n+1}^+ vanish).
3. It has at least first-order zeroes at the points P_k^- , $k = 1, \dots, m$, Q_k^- , $k = 1, \dots, n$ and R_k , $k = 1, \dots, l + 1$.

From the Riemann-Roch theorem it follows that, for generic data, this function is identically equal to 0.

Remark. In the paper [18] the points P_k^+ , P_k^- , Q_k^+ , Q_k^- were generic. The special case in which all the P_k^+ 's coincide: $P_k^+ = P^+$ for $k = 1, \dots, M$, as well as all the P_k^- 's, Q_k^+ 's, Q_k^- 's: $P_k^- = P^-$, $k = 1, \dots, M$, $Q_k^+ = Q^+$, $Q_k^- = Q^-$, $k = 1, \dots, N$ was also discussed in [18]. In this case, the discrete time shift $m \rightarrow m + 1$ corresponds to increasing the order of the pole at the point P^+ and the order of the zero at the point P^- of one, and the Abel transform in the θ -functional formulas (see below) becomes constant. This last choice results in an essential effectivization of the explicit formulas, but it corresponds to a more restricted class of potentials. For example, if the potentials are periodic in m with period M , the Floquet multiplier at the point Q_{n+1}^- can be calculated by the following formula

$$\kappa_1(n) = (-1)^M \prod_{m=1}^M \frac{\alpha_3(m, n)}{\alpha_1(m, n)}.$$

If all points Q_n^- coincide, then $\kappa_1(n)$ should not depend on n .

Since we do not like to impose such a restriction on the class of our potentials, we assume a generic collections of poles and zeroes.

5.1 Discrete Moutard reductions and constraints on the spectral data

Let us describe the reductions corresponding to the affine 4-point scheme (20) (may be with a complex potential $f(m, n)$).

Lemma 16. *Assume that Γ possess a holomorphic involution σ with exactly 2 fixed points $R_+ = R_1$, R_- . Assume that the spectral data have the following symmetry with respect to σ :*

1. *Exactly one normalization point $R_1 = R_+$ is used ($l = 0$).*
2. *There exists a meromorphic differential Ω with 2 first-order poles at the fixed points R_+ , R_- and $2g$ zeroes at the points $\gamma_1, \dots, \gamma_g, \sigma\gamma_1, \dots, \sigma\gamma_g$.*
3. *$\sigma P_k^+ = P_k^-$, $\sigma Q_k^+ = Q_k^-$ for all k .*

Then

$$\alpha_3(m, n) + 1 = 0, \quad \alpha_1(m, n) + \alpha_2(m, n) = 0 \quad (113)$$

and, consequently, $\Psi(\gamma, m, n)$ satisfies the 4-point scheme (20) with

$$f(m, n) = i\alpha_1(m, n) = -i \lim_{\gamma \rightarrow P_{m+1}^+} \frac{\Psi(\gamma, m+1, n+1)}{\Psi(\gamma, m+1, n)} = i \lim_{\gamma \rightarrow Q_{n+1}^+} \frac{\Psi(\gamma, m+1, n+1)}{\Psi(\gamma, m, n+1)}. \quad (114)$$

Proof. By analogy with [30], consider the following form:

$$\Omega\Psi(\gamma, m, n)\Psi(\sigma\gamma, m, n).$$

This form has 2 first-order poles at the points R_+ , R_- and no other singularities. Therefore

$$\text{res}(\Omega\Psi(\gamma, m, n)\Psi(\sigma\gamma, m, n), \gamma = R_+) = -\text{res}(\Omega\Psi(\gamma, m, n)\Psi(\sigma\gamma, m, n), \gamma = R_-)$$

Taking into account that

$$\text{res}(\Omega, \gamma = R_+) = -\text{res}(\Omega, \gamma = R_-),$$

we obtain

$$\Psi^2(R_-, m, n) = \Psi^2(R_+, m, n) \equiv 1.$$

Let us show that

$$\Psi(R_-, m, n) = (-1)^{m+n}. \quad (115)$$

If $m = n = 0$ we know that $\Psi(\gamma, 0, 0) \equiv 1$ (from the uniqueness argument and taking into account that the constant 1 satisfies all the analyticity constraints). Therefore $\Psi(R_-, 0, 0) = 1$. Consider the function $\Psi(R_-, m+1, n)$ as a function of the point P_{m+1}^+ (we use here that $P_{m+1}^- = \sigma P_{m+1}^+$). By construction this function is meromorphic in P_{m+1}^+ . Since the surface Γ is connected, then $\Psi(R_-, m+1, n)$ does not depend on P_{m+1}^+ . Assume that P_{m+1}^+ be located very close to R_- and denote by z a local coordinate in a neighbourhood of R_- , $z(R_-) = 0$, such that $\sigma z = -z$. Then

$$\Psi(z, m+1, n) = \frac{z + P_{m+1}^+}{z - P_{m+1}^+} \Psi(z, m, n) + o(P_{m+1}^+);$$

therefore

$$\Psi(R_-, m+1, n) = -\Psi(R_-, m, n).$$

Substituting (115) into (109) and taking into account (112), we obtain (113). This completes the proof. \square

Having in mind applications in discrete geometry and in discrete complex functions theory, let us formulate sufficient conditions which guaranty that the potential $f(m, n)$ be real.

Lemma 17. *Assume that all the constraints of Lemma 16 on the spectral data be fulfilled. Let the spectral data satisfy the following additional restrictions. There exists an antiholomorphic involution τ on Γ with the following properties:*

1. τ commutes with σ .
2. $\tau R_+ = R_-$.
3. The points P_k^+ , P_k^- , Q_k^+ , Q_k^- are fixed point of τ .
4. The set of divisor points γ_k is invariant under τ (but τ may map one divisor point into another).

Then the potential $f(m, n)$ and the function $\Psi(\gamma, m, n)$ have the following reality properties:

$$\bar{f}(m, n) = f(m, n), \quad \Psi(\tau\gamma, m, n) = (-1)^{m+n} \overline{\Psi(\gamma, m, n)}. \quad (116)$$

In particular, if γ is a real point of Γ : $\tau\gamma = \gamma$, then $\Psi(\gamma, m, n)$ is either real or pure imaginary:

$$\begin{aligned} \Psi(\gamma, m, n) &\in \mathbb{R} && \text{if } m+n \text{ is even} \\ \Psi(\gamma, m, n) &\in i\mathbb{R} && \text{if } m+n \text{ is odd.} \end{aligned} \quad (117)$$

To prove the Lemma, it is sufficient to point out that the function $(-1)^{m+n}\overline{\Psi(\tau\gamma, m, n)}$ satisfies all the analytic constraints characterizing $\Psi(\gamma, m, n)$. Therefore these two functions coincide. The reality of $f(m, n)$ then follows immediately from (114).

Remark. The reality constraint on the eigenfunction $\Psi(\gamma, m, n)$ naturally distinguishes between even and odd sub-lattices.

In Section 2 it was shown on the algebraic level that any solution of the 4-point scheme (20) generates a solution of the 5-point scheme (11). In this section we give an analytic proof of this statement.

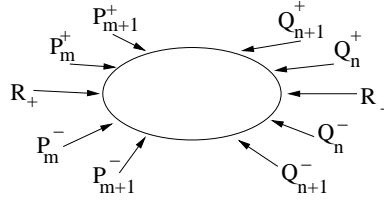
Consider the following meromorphic 1-form

$$\tilde{\Omega}(\gamma, m, n, \tilde{m}, \tilde{n}) = \Omega(\gamma)\Psi(\sigma\gamma, m, n)\Psi(\gamma, \tilde{m}, \tilde{n}).$$

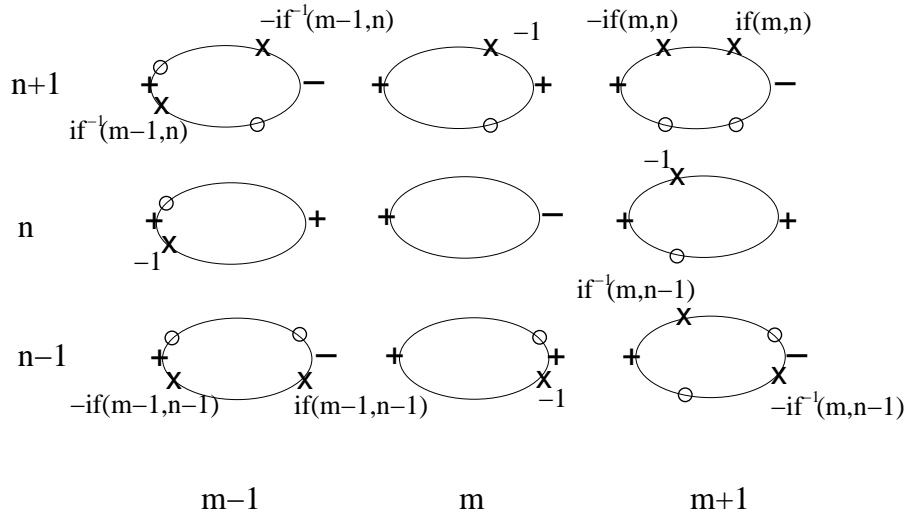
Assume that the differential Ω used before to define the constraints on the spectral data has the following normalization:

$$\text{res}(\Omega, \gamma = R_+) = \frac{1}{2}, \quad \text{res}(\Omega, \gamma = R_-) = -\frac{1}{2}.$$

Let us calculate the residues at the poles of $\tilde{\Omega}(\gamma, m, n, \tilde{m}, \tilde{n})$ for $|\tilde{m} - m| \leq 1$, $|\tilde{n} - n| \leq 1$. It is convenient to write the answer in graphic form. We denote our Riemann surface by an oval, and we assume that our distinguished points are located at the following positions:



In the next picture we show the poles of $\tilde{\Omega}(\gamma, m, n, \tilde{m}, \tilde{n})$. We use the following notations: + denotes a pole with residue $+\frac{1}{2}$, - denotes a pole with residue $-\frac{1}{2}$, \times denotes a pole with a residue different from $\pm\frac{1}{2}$, \circ denotes a zero. The parameter \tilde{n} is written in the left column, the parameter \tilde{m} is written in the bottom row. The residues are written next to the symbols \times .



Let us explain how the residues are calculated. First of all, for $\tilde{m} = m, \tilde{n} = n$ we have only 2 poles and the residues are opposite, therefore we have an identity (which was used when we proved the Lemma 16). If $\tilde{m} = m \pm 1, \tilde{n} = n$ or $\tilde{m} = m, \tilde{n} = n \pm 1$ we have 3 poles. At R_+ and R_- the residues are $+\frac{1}{2}$, therefore the 3rd residue is -1 . If $|\tilde{m} - m| = |\tilde{n} - n| = 1$, the residue at R_+ is equal to $+\frac{1}{2}$, the residue at R_- is equal to $-\frac{1}{2}$, therefore we have 2 more poles with opposite residues. For example, let us calculate the residues for $\tilde{m} = m + 1, \tilde{n} = n + 1$. From the 4-point scheme (20) it follows that

$$\text{res}(\tilde{\Omega}(\gamma, m, n, m + 1, n + 1), \gamma = P_{m+1}^+) = if(m, n) \text{res}(\tilde{\Omega}(\gamma, m, n, m + 1, n), \gamma = P_{m+1}^+),$$

therefore

$$\text{res}(\tilde{\Omega}(\gamma, m, n, m + 1, n + 1), \gamma = P_{m+1}^+) = -if(m, n). \quad (118)$$

All the other residues can be calculated exactly in the same way.

To prove that the function $\Psi(\gamma, m, n)$ satisfies the 5-point scheme

$$\begin{aligned} & \frac{1}{f(m, n)}(\Psi(\gamma, m + 1, n + 1) - \Psi(\gamma, m, n)) + \\ & \frac{1}{f(m - 1, n - 1)}(\Psi(\gamma, m - 1, n - 1) - \Psi(\gamma, m, n)) + \\ & f(m, n - 1)(\Psi(\gamma, m + 1, n - 1) - \Psi(\gamma, m, n)) + \\ & f(m - 1, n)(\Psi(\gamma, m - 1, n + 1) - \Psi(\gamma, m, n)) = 0 \end{aligned} \quad (119)$$

it is enough to verify that the left-hand side of (119) has the poles and zeroes prescribed for $\Psi(\gamma, m, n)$ and, moreover it is equal to zero at the points R_+, R_- . Therefore, by the Riemann-Roch theorem, it is zero.

5.2 Algebro-geometrical solutions

Riemann surfaces with the constraints described above can be constructed in the following way. We start from the “vacuum” solution, corresponding to the Riemann sphere $g = 0$:

$$if(m, n) = \frac{P_{m+1} + Q_{n+1}}{P_{m+1} - Q_{n+1}}, \quad \Psi(\lambda, m, n) = \prod_{k=1}^m \frac{\lambda + P_k}{\lambda - P_k} \prod_{k=1}^n \frac{\lambda + Q_k}{\lambda - Q_k}.$$

If all P_k ’s coincide as well as all Q_k ’s: $P_k = P, Q_k = Q$, the above vacuum solution reduces to the so-called “discrete exponential”

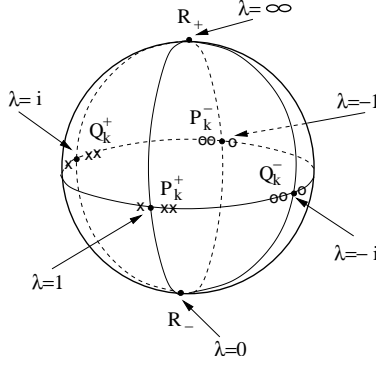
$$e(m, n; \lambda) = \left(\frac{\lambda + P}{\lambda - P} \right)^m \left(\frac{\lambda + Q}{\lambda - Q} \right)^n \quad (120)$$

discussed in [4]; if $Q = iP$, then $f(m, n) \equiv 1$ and (120) solves the Cauchy-Riemann equation (9).

The involutions σ and τ are given by:

$$\sigma\lambda = -\lambda, \quad \tau\lambda = \frac{1}{\lambda}$$

and the Riemann sphere with all its distinguished points is drawn here:



By analogy with the continuous case, discrete holomorphic polynomials can be constructed from the discrete exponential (120) in the following way (if $Q = iP$):

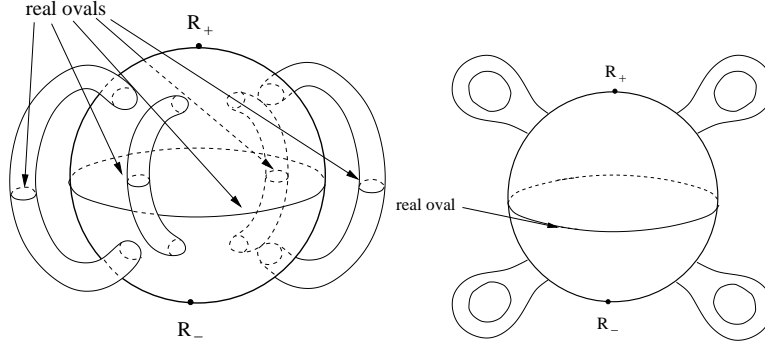
$$p^{(j)}(m, n) = \frac{1}{(2P)^j} \left. \frac{d^j e(m, n; 1/\zeta)}{d\zeta^j} \right|_{\zeta=0}. \quad (121)$$

The first 3 examples read:

$$p^{(0)} = 1, \quad p^{(1)} = m + in, \quad p^{(2)} = (m + in)^2, \quad p^{(3)} = (m + in)^3 + (m - in)/2. \quad (122)$$

We observe that the discrete holomorphic polynomials coincide with the continuous monomials z^j (assuming $z = m + in$) up to the order 2.

A non-trivial Riemann surface can be constructed by attaching handles to the above Riemann sphere keeping the above symmetries. Two examples are shown below.



The figure on the left illustrates the so-called M -curve (the number of real ovals, i.e. the ovals formed by the fixed points of the antiholomorphic involution τ , is equal $g + 1$. $g + 1$ is the greatest possible number of real ovals. In this particular case $g = 4$). The figure on the right illustrates a curve with $g = 4$ and only one real oval.

Remark. In the theory of the 2-D continuous Schrödinger operator the M -curves play a distinguished role. In particular the operators generated by the M -curves are non-singular and strictly positive [30], [14]. A complete classification of finite-gap data corresponding to nonsingular solutions was obtained in [23]. The study of analogous properties of the 4-point and 5-point schemes will be the subject of future investigation.

It is well-known that function $\Psi(\gamma, m, n)$ can be written in terms of Riemann θ -functions in the following way:

$$\begin{aligned}
\Psi(\gamma, m, n) = & \frac{\theta \left(\vec{A}(\gamma) + \sum_{k=1}^m (\vec{A}(P_k^-) - \vec{A}(P_k^+)) + \sum_{k=1}^n (\vec{A}(Q_k^-) - \vec{A}(Q_k^+)) - \sum_{k=1}^g \vec{A}(\gamma_k) - \vec{K} \middle| B \right)}{\theta \left(\vec{A}(\gamma) - \sum_{k=1}^g \vec{A}(\gamma_k) - \vec{K} \middle| B \right)} \times \\
& \times \frac{\theta \left(\vec{A}(R_+) - \sum_{k=1}^g \vec{A}(\gamma_k) - \vec{K} \middle| B \right)}{\theta \left(\vec{A}(R_+) + \sum_{k=1}^m (\vec{A}(P_k^-) - \vec{A}(P_k^+)) + \sum_{k=1}^n (\vec{A}(Q_k^-) - \vec{A}(Q_k^+)) - \sum_{k=1}^g \vec{A}(\gamma_k) - \vec{K} \middle| B \right)} \times \\
& \times \exp \left(\sum_{k=1}^m \int_{R_+}^{\gamma} \Omega(\tilde{\gamma}, P_k^+, P_k^-) + \sum_{k=1}^n \int_{R_+}^{\gamma} \Omega(\tilde{\gamma}, Q_k^+, Q_k^-) \right).
\end{aligned} \tag{123}$$

Here we have used the following notations. a_k, b_k are the basic cycles in Γ , $a_k \circ b_l = \delta_{kl}$, $a_k \circ a_l = b_k \circ b_l = 0$. ω_i are the basic holomorphic differentials such that

$$\oint_{a_j} \omega_i = \delta_{ij}, \tag{124}$$

$\Omega(\gamma, P, Q)$ are meromorphic differentials of the third kind with 2 first-order poles in P, Q , with residues -1 and $+1$ respectively and zero a -periods:

$$\text{res}(\Omega(\gamma, P, Q), \gamma = P) = -1, \quad \text{res}(\Omega(\gamma, P, Q), \gamma = Q) = 1, \quad \oint_{a_j} \Omega(\gamma, P, Q) = 0. \tag{125}$$

$\vec{A}(\gamma) = (A_1(\gamma), \dots, A_g(\gamma))$ denotes the Abel transform:

$$A_k(\gamma) = \int_{\mathcal{P}}^{\gamma} \omega_k, \quad k = 1, \dots, g, \tag{126}$$

where the starting point \mathcal{P} of the Abel transform can be chosen arbitrarily. The Riemann θ -functions are defined by the following Fourier series:

$$\theta(\vec{z}|B) = \sum_{m_1, \dots, m_g} \exp \left\{ \pi i \sum_{k,j} B_{kj} m_k m_j + 2\pi i \sum_k z_k m_k \right\}, \tag{127}$$

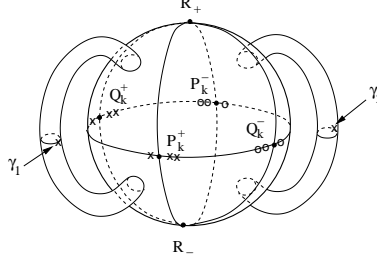
where B_{kl} are the b -periods of the holomorphic differentials:

$$\oint_{b_j} \omega_i = B_{ji} \tag{128}$$

(in the definition of the θ -function we use the same normalizations as in [20]; other classical books, like [12], use the different normalization corresponding to a-periods equal to $2\pi i$).

Due to the symmetry of the spectral curve imposed by σ , the Riemann θ -functions can be expressed as bilinear combinations of Riemann and Prym θ -functions of genus $g/2$ [12]. This is analogous to the continuous Schrödinger case [30].

The simplest non-trivial example corresponds to an M -curve with 2 handles.



In this case, due to the symmetry imposed by σ , the answer can be expressed in terms of genus 1 θ -functions. It is convenient to choose a basis of cycles with the following symmetry (see [12]):

$$\sigma a_1 = -a_2, \quad \sigma b_1 = -b_2. \quad (129)$$

Denote by ω and $\hat{\omega}$ the holomorphic differentials such that

$$\sigma \omega = \omega, \quad \sigma \hat{\omega} = -\hat{\omega}, \quad \oint_{a_1} \omega = \oint_{a_1} \hat{\omega} = 1 \quad (130)$$

and let

$$\eta = \oint_{b_1} \omega, \quad \hat{\eta} = \oint_{b_1} \hat{\omega}. \quad (131)$$

Then

$$\omega_1 = \frac{\hat{\omega} + \omega}{2}, \quad \omega_2 = \frac{\hat{\omega} - \omega}{2}, \quad B_{11} = B_{22} = \frac{\hat{\eta} + \eta}{2}, \quad B_{12} = B_{21} = \frac{\hat{\eta} - \eta}{2}, \quad (132)$$

and

$$\begin{aligned} \theta(z_1, z_2 | B) &= \theta(z_1 + z_2 | 2\eta) \theta(z_1 - z_2 | 2\hat{\eta}) + \\ &\theta(z_1 + z_2 + \eta | 2\eta) \theta(z_1 - z_2 + \hat{\eta} | 2\hat{\eta}) \exp \left[\pi i \left(\frac{\eta + \hat{\eta}}{2} + 2z_1 \right) \right]. \end{aligned} \quad (133)$$

Let us consider the following explicit example in which the Riemann surface Γ is defined by the equation:

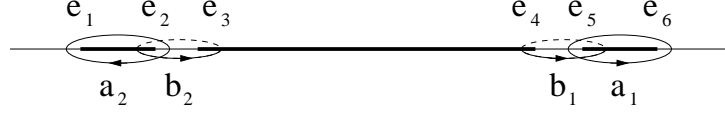
$$\mu^2 = \prod_{k=1}^6 (\lambda - e_k), \quad e_{7-k} = -e_k, \quad (134)$$

R_+ , R_- are the infinite points $\mu \sim \lambda^3$ and $\mu \sim -\lambda^3$ respectively, with $\lambda \sim \infty$, $\sigma : (\lambda, \mu) \rightarrow (-\lambda, -\mu)$, $\tau : (\lambda, \mu) \rightarrow (\bar{\lambda}, -\bar{\mu})$, ω and $\hat{\omega}$ are defined by:

$$\omega = d_1 \frac{d\lambda}{i\mu}, \quad \hat{\omega} = d_2 \frac{\lambda d\lambda}{i\mu}, \quad (135)$$

where the parameters d_1 , d_2 are fixed by the normalization condition (130). The a and b cycles are shown in the figure below, the cuts are along the intervals $[e_1, e_2]$, $[e_3, e_4]$, $[e_5, e_6]$,

the solid lines correspond to the sheet containing R_+ , the dashed lines correspond to the sheet containing R_- .



The divisor is located on the ovals a_1, a_2 , the distinguished points $P_k^+, P_k^-, Q_k^+, Q_k^-$, are located on the oval lying over the interval $[e_3, e_4]$.

If the starting point of the Abel transform is chosen at one of the branch points of a hyperelliptic Riemann surface, the vector of Riemann constants can be written in a very simple form (see [12]). If $\mathcal{P} = e_6$, then

$$\vec{K} = \left[\begin{array}{c} \frac{1}{2} + \frac{B_{12}}{2} \\ \frac{1}{2} + \frac{B_{22}}{2} \end{array} \right]. \quad (136)$$

5.3 Some applications

As it was already mentioned in the introduction, the 4-point and 5-point schemes (20) and (11) are relevant in the theory of discrete complex functions, as well as in the theory of integrable discrete geometries.

Solutions $\Psi(\gamma, m, n)$ of equation (20) provide examples of discrete holomorphic functions and, through the change of variables (25), generate discrete harmonic functions on the sub-lattice. Let us concentrate here our attention, for instance, on the discrete analogues of a polynomial of degree 2.

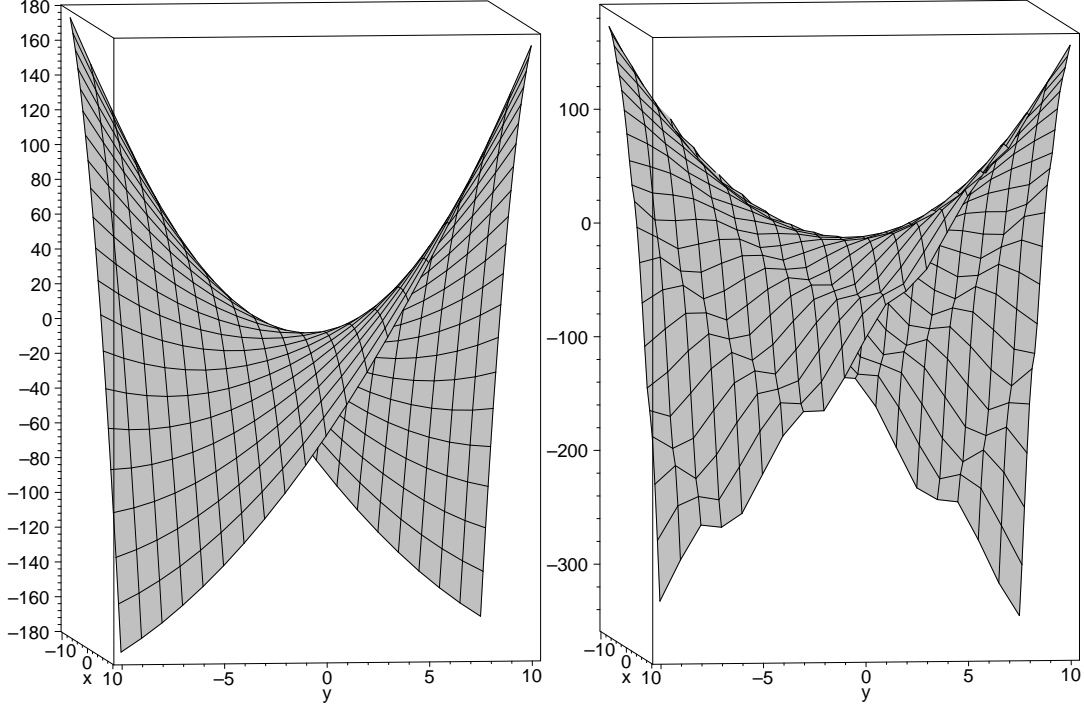
If $g = 2$, the corresponding discrete holomorphic function is constructed from the $\Psi(\gamma, m, n)$ function (123),(127) associated with the hyperelliptic curve (134), via the formula:

$$\lim_{\gamma \rightarrow R_+} \left([\Psi(\gamma, m, n) + \Psi(\sigma\gamma, m, n) - 2\Psi(R_+, m, n)] \frac{\lambda^2}{4} \right). \quad (137)$$

Due to the positions of the distinguished points P^+, Q^+ , the corresponding $g = 0$ discrete holomorphic function has the following form:

$$\Re \left(e^{i\frac{4}{3}\pi} p^{(2)}(m, n) \right) = \Re \left(e^{i\frac{4}{3}\pi} (m + in)^2 \right). \quad (138)$$

The graphs of the real parts of the discrete holomorphic functions (138) and (137) are shown in the figure below.



$g = 0$

$g = 2$

For $g = 2$ the spectral data are: $e_1 = -3$, $e_2 = -2.02$, $e_3 = -2$, $\gamma_1 = (2.6, i\sqrt{16.56635904})$, $\gamma_2 = (-2.6, i\sqrt{16.56635904})$, $P^+ = (-1, i\sqrt{73.9296})$, $Q^+ = (-\sqrt{3}, -i\sqrt{6.4824})$.

As it was shown in [25], solutions $\Psi(\gamma, \mu, \nu)$ of equation (11) allow to construct quadrilateral surfaces in \mathbb{R}^3 , i.e., discrete surfaces whose elementary quadrilaterals are planar, in the following way.

Consider the vector $\vec{N}_{\mu, \nu} : \mathbb{Z}^2 \rightarrow \mathbb{R}^3$ whose components are three independent real solutions of the self-adjoint 5-point scheme (11). Then $\vec{N}_{\mu, \nu}$ is the normal vector of a quadrilateral surface $\vec{r}_{\mu, \nu} : \mathbb{Z}^2 \rightarrow \mathbb{R}^3$ defined by the following generalization of the Lelievre formulas:

$$\Delta_\mu \vec{r}_{\mu, \nu} = -b_{\mu, \nu-1} \vec{N}_{\mu, \nu} \times \vec{N}_{\mu, \nu-1}, \quad \Delta_\nu \vec{r}_{\mu, \nu} = a_{\mu-1, \nu} \vec{N}_{\mu, \nu} \times \vec{N}_{\mu-1, \nu}. \quad (139)$$

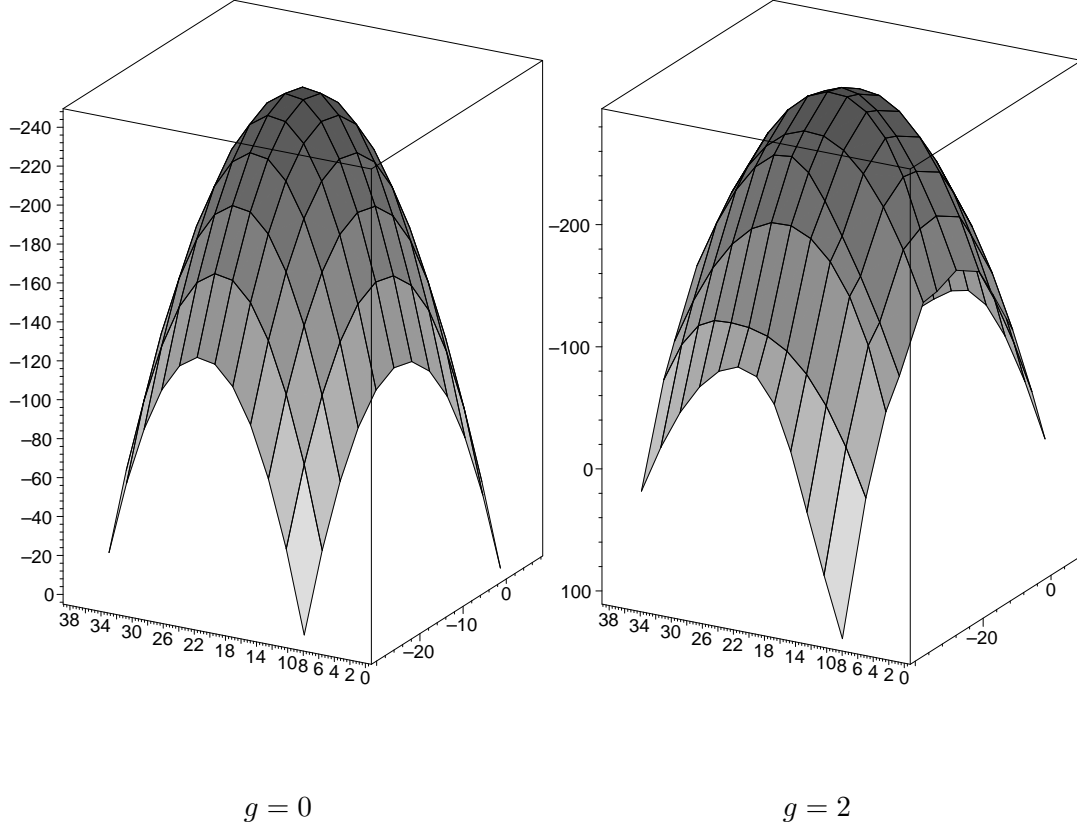
Here we consider examples of quadrilateral surfaces constructed using the discrete analogues of complex polynomials of degree 1. If $g = 0$, we consider the following normal vector:

$$\vec{N}_{\mu, \nu}^{(0)} = \left[2\Re \left(e^{i\frac{2}{3}\pi} (\mu - \nu + i(\mu + \nu)) \right), 2\Im \left(e^{i\frac{2}{3}\pi} (\mu - \nu + i(\mu + \nu)) \right), 1 \right]. \quad (140)$$

Analogously, if $g = 2$, the normal vector is defined by

$$\vec{N}_{\mu, \nu}^{(2)} = \left[\Re \left(\lim_{\gamma \rightarrow R_+} ([\Psi(\gamma, \mu - \nu, \mu + \nu) - 1] \lambda) \right), \Im \left(\lim_{\gamma \rightarrow R_+} ([\Psi(\gamma, \mu - \nu, \mu + \nu) - 1] \lambda) \right), 1 \right], \quad (141)$$

where Ψ is the wave function (123),(127) associated with the hyperelliptic curve (134). Then the two quadrilateral lattices constructed, via the embedding (139), by the normal vectors $\vec{N}_{\mu,\nu}^{(0)}$ and $\vec{N}_{\mu,\nu}^{(2)}$, are shown in the figures below.



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